GRADED MAXIMAL COHEN-MACAULAY MODULES OVER NONCOMMUTATIVE GRADED GORENSTEIN ISOLATED SINGULARITIES

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Abstract. In this paper, we define a notion of noncommutative graded isolated singularity, and study AS-Gorenstein isolated singularities and the categories of graded maximal Cohen-Macaulay modules over them. In particular, for an AS-Gorenstein algebra A of dimension $d \geq 2$, we show that A is a graded isolated singularity if and only if the stable category of graded maximal Cohen-Macaulay modules over A has the Serre functor. Using this result, we also show the existence of cluster tilting objects in the categories of graded maximal Cohen-Macaulay modules over Veronese subalgebras of certain AS-regular algebras.

1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic 0. In representation theory of orders, which generalize both finite dimensional algebras and Cohen-Macaulay rings, studying the categories of Cohen-Macaulay modules is active (see [8] for details). In particular, the following results play key roles in the theory (we present graded versions due to [9, Corollary 3.5, Theorem 4.2, Theorem 5.2]).

Theorem 1.1. *Let R be a noetherian commutative graded local Gorenstein ring of dimension d and of Gorenstein parameter ℓ. Assume that R is an isolated singularity. Then the stable category of graded maximal Cohen-Macaulay modules has the Serre functor $(-\ell)[d-1]$.*

Theorem 1.2. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring generated in degree 1, G a finite subgroup of $\mathrm{SL}_d(k)$ acting linearly on S, and S^G the fixed subring of S.*

1. *Then the skew group algebra $S * G$ is isomorphic to $\underline{\mathrm{End}}_{S^G}(S)$ as graded algebras.*
2. *Assume that S^G is an isolated singularity. Then S is a $(d-1)$-cluster tilting object in the category of graded maximal Cohen-Macaulay modules over S^G.*

The proofs of these results rely on commutative ring theory. This paper tries to give a noncommutative (not necessarily order) version of them.

One of the noncommutative analogues of polynomial rings (resp. Gorenstein local rings) is AS-regular algebras (resp. AS-Gorenstein algebras). In this paper, we define a notion of noncommutative graded isolated singularity by the smoothness of the noncommutative projective scheme (see also [13]), and we focus on studying AS-Gorenstein isolated singularities.

The first of the main results is as follows (see Corollary 4.5):

Theorem 1.3. *Let A be a (noetherian) AS-Gorenstein algebra of dimension $d \geq 2$ with the canonical module ω_A. Then the following are equivalent.*

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(1) *A is a graded isolated singularity.*
(2) *The bounded derived category of noncommutative projective scheme associated to A has the Serre functor* $- \otimes^{\mathrm{L}}_{A} \omega_A[d-1]$.
(3) *The stable category of graded maximal Cohen-Macaulay modules over A has the Serre functor* $- \otimes_A \omega_A[d-1]$.

If R is a noetherian commutative graded local Gorenstein ring of dimension d and of Gorenstein parameter ℓ, then $\omega_R \cong R(-\ell)$ as graded R-R bimodules, so this is a generalization of Theorem 1.1. The stable category of graded maximal Cohen-Macaulay modules is one of the main objects in representation theory, while the bounded derived category of noncommutative projective scheme is one of the main objects of study in noncommutative algebraic geometry. Theorem 1.3 gives an explicit connection between their Serre functors.

Using Theorem 1.3, we can get the following second main result (see Corollary 5.12):

Theorem 1.4. *Let A be a (noetherian) AS-regular domain of dimension $d \geq 2$ and of Gorenstein parameter ℓ generated in degree 1. Take $r \in \mathbb{N}^+$ such that $r \mid \ell$. We define a graded algebra automorphism σ_r of A by $\sigma_r(a) = \xi^{\deg a} a$ where ξ is a primitive r-th root of unity, and write $G = \langle \sigma_r \rangle$ for the finite cyclic subgroup of* $\mathrm{GrAut}(A)$ *generated by σ_r. Then*

(1) *the skew group algebra $A * G$ is isomorphic to $\underline{\mathrm{End}}_{A^G}(A)$ as graded algebras.*
(2) *A^G is a graded isolated singularity, and A is a $(d-1)$-cluster tilting object in the category of graded maximal Cohen-Macaulay modules over A^G.*

Note that A^G is the r-th Veronese subalgebra with the grading as a graded subalgebra of A. This is a partial generalization of Theorem 1.2. Thanks to this result, we can obtain many examples of $(d-1)$-cluster tilting objects in the category of graded maximal Cohen-Macaulay modules over non-orders.

2. Preliminaries

Let A be a connected graded k-algebra. We denote by $\mathsf{Gr}\, A$ the category of graded right A-modules with degree zero A-module homomorphisms, and by $\mathsf{gr}\, A$ the full subcategory consisting of finitely generated graded right A-modules. The group of graded k-algebra automorphisms of A is denoted by $\mathrm{GrAut}\, A$.

Let M be a graded right A-module. For an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \mathsf{Gr}\, A$ and the shift $M(n) \in \mathsf{Gr}\, A$ by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. Note that the rule $M \mapsto M(n)$ is a k-linear autoequivalence for $\mathsf{Gr}\, A$ and $\mathsf{gr}\, A$, called the shift functor. We write

$$\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathsf{Gr}\, A}(M, N(n)).$$

For a graded algebra automorphism $\sigma \in \mathrm{GrAut}\, A$, we define a new graded right A-module $M_\sigma \in \mathsf{Gr}\, A$ by $M_\sigma = M$ as graded vector spaces with the new right action $m * a = m\sigma(a)$ for $m \in M$ and $a \in A$. We denote by $(-)^* = \underline{\mathrm{Hom}}_k(-, k)$ the graded Matlis duality. If M is locally finite, then $M^{**} \cong M$ as graded A-modules. We denote by $(-)^\vee$ both $\underline{\mathrm{Hom}}_A(-, A)$ and $\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(-, A)$. For $M \in \mathsf{Gr}\, A$, take a minimal free resolution

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to M \to 0\ .$$

The n-th syzygy module $\Omega^n M$ of M is defined by $\mathrm{Im}\, d_n$. Applying $(-)^\vee$, we define the transpose $\mathrm{Tr}\, M$ of M by the exact sequence

$$0 \to M^\vee \to F_0^\vee \xrightarrow{d_1^\vee} F_1^\vee \to \mathrm{Tr}\, M \to 0.$$

Let C be an abelian category. We denote by $\mathsf{D}(\mathsf{C})$ the derived category of C. The bounded derived category of C is denoted by $\mathsf{D}^{\mathsf{b}}(\mathsf{C})$.

Let A be a noetherian connected graded algebra with the unique maximal homogeneous ideal $\mathfrak{m} = A_{\geq 1}$. The graded bimodule $A/\mathfrak{m}$ is one-dimensional over k, and it is usually denoted simply by k. We define the functor $\underline{\Gamma}_{\mathfrak{m}} : \mathsf{Gr}\, A \to \mathsf{Gr}\, A$ by $\underline{\Gamma}_{\mathfrak{m}}(-) = \lim_{n\to\infty} \underline{\mathrm{Hom}}_A(A/A_{\geq n}, -)$. The derived functor of $\underline{\Gamma}_{\mathfrak{m}}$ is denoted by $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-)$, and its cohomologies are denoted by $\underline{\mathrm{H}}^i_{\mathfrak{m}}(-) = h^i(\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-))$. For $M \in \mathsf{Gr}\, A$, we define the following numbers:

$$\operatorname{depth}_A M = \inf \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(M) = \inf\{i \mid \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \neq 0\},$$
$$\operatorname{ldim}_A M = \sup \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(M) = \sup\{i \mid \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \neq 0\}.$$

Note that for any $M \in \mathsf{Gr}\, A$,

$$\operatorname{depth}_A M = \inf \mathrm{R}\underline{\mathrm{Hom}}_A(k, M) = \inf\{i \mid \underline{\mathrm{Ext}}^i_A(k, M) \neq 0\}.$$

Let A be a noetherian connected graded algebra with a balanced dualizing complex D (see [22, Definitions 3.3, 4.1]). We say that A is balanced Cohen-Macaulay of depth d if there exists a graded A-A bimodule ω_A such that $D \cong \omega_A[d]$. We call this graded A-A bimodule ω_A the canonical module of A. Note that if A is balanced Cohen-Macaulay of depth d, then there is an isomorphism $\omega_A = \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)^*$ in $\mathsf{Gr}\, A^e$, and for every $M \in \mathsf{Gr}\, A$,

$$\mathrm{R}\underline{\mathrm{Hom}}_A(M, \omega_A) \cong \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(M)^*[-d]$$

in $\mathsf{D}(\mathsf{Gr}\, A^{\mathrm{op}})$ by [21, Theorem 5.1]. Let A be a balanced Cohen-Macaulay algebra. We say that $M \in \mathsf{gr}\, A$ is (graded) maximal Cohen-Macaulay if $\operatorname{depth}_A M = \operatorname{ldim}_A M = \operatorname{depth}_A A < \infty$. We denote by $\mathsf{CM}^{\mathsf{gr}}(A)$ the full subcategory of $\mathsf{gr}\, A$ consisting of maximal Cohen-Macaulay modules.

Definition 2.1. A connected graded algebra A is called a d-dimensional AS-Gorenstein algebra (resp. AS-regular algebra) of Gorenstein parameter ℓ if

- A is noetherian,
- $\operatorname{id}_A A = \operatorname{id}_{A^{\mathrm{op}}} A = d < \infty$ (resp. $\operatorname{gldim} A = d < \infty$) and
- $\underline{\mathrm{Ext}}^i_A(k, A) \cong \underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

By [11, Theorem 1.2], if A is a d-dimensional AS-Gorenstein algebra of Gorenstein parameter ℓ, then A has a balanced dualizing complex $A_\nu(-\ell)[d]$ for some graded algebra automorphism $\nu \in \operatorname{GrAut} A$, so A is balanced Cohen-Macaulay. We call this $\nu \in \operatorname{GrAut} A$ the generalized Nakayama automorphism of A.

In this paper,"an AS-Gorenstein algebra" means "a d-dimensional AS-Gorenstein algebra of Gorenstein parameter ℓ with the generalized Nakayama automorphism $\nu \in \operatorname{GrAut} A$ and the canonical module ω_A".

Let A be a noetherian connected graded algebra. We say that A satisfies χ if $\underline{\mathrm{Ext}}^i_A(k, M)$ are finite dimensional over k for all $M \in \mathsf{gr}\, A$ and all i. For details, we refer to [2]. The relationship between χ and the existence of the balanced dualizing complex is given by [21, Theorem 6.3].

We denote by $\mathsf{tors}\, A$ the full subcategory of $\mathsf{gr}\, A$ consisting of finite dimensional modules over k, and

$$\mathsf{tails}\, A := \mathsf{gr}\, A/\mathsf{tors}\, A$$

the quotient category, which is called the noncommutative projective scheme associated to A in [2]. If A is a commutative graded algebra finitely generated in degree 1 over k, then $\mathsf{tails}\, A$ is equivalent to the category of coherent sheaves on $\operatorname{Proj} A$ by results of Serre, justifying the terminology. We usually denote by $\mathcal{M} \in \mathsf{tails}\, A$ the image of $M \in \mathsf{gr}\, A$. Note that the k-linear autoequivalence $M \mapsto M(n)$ preserves finite dimensional modules over

k, so it induces a k-linear autoequivalence $\mathcal{M} \mapsto \mathcal{M}(n)$ for $\mathsf{tails}\, A$, again called the shift functor. We also use the notation

$$\underline{\mathrm{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_{\mathsf{tails}\, A}(\mathcal{M}, \mathcal{N}(n)).$$

By [2, Proposition 7.2 (1)], we have a functorial isomorphism

$$\underline{\mathrm{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_A(M_{\geq n}, N) \tag{2.1}$$

for $M, N \in \mathsf{gr}\, A$.

We define a notion of noncommutative graded isolated singularity by the smoothness of the noncommutative projective scheme. Recall that the global dimension of $\mathsf{tails}\, A$ is defined by

$$\mathrm{gldim}(\mathsf{tails}\, A) := \sup\{i \mid \mathrm{Ext}^i_{\mathsf{tails}\, A}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \mathsf{tails}\, A\}.$$

Definition 2.2. A noetherian connected graded algebra A is called a graded isolated singularity if $\mathsf{tails}\, A$ has finite global dimension.

If A is a graded quotient of a polynomial ring generated in degree 1, then A is a graded isolated singularity (in the above sense) if and only if $A_{(\mathfrak{p})}$ is regular for any homogeneous prime ideal $\mathfrak{p} \neq \mathfrak{m}$, justifying the definition. It is easy to see that if A has finite global dimension, then $\mathsf{tails}\, A$ has finite global dimension, so A is a graded isolated singularity. The purpose of this paper is to study AS-Gorenstein isolated singularities.

The B-construction defined below is often used in noncommutative algebraic geometry. We refer to [17] for details.

Definition 2.3. An algebraic triple $(\mathsf{C}, \mathcal{O}, s)$ over k consists of a k-linear category C, an object $\mathcal{O} \in \mathsf{C}$, and a k-linear autoequivalence $s \in \mathrm{Aut}\, \mathsf{C}$. We define the graded algebra associated to an algebraic triple $(\mathsf{C}, \mathcal{O}, s)$ by

$$B(\mathsf{C}, \mathcal{O}, s) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{C}}(\mathcal{O}, s^i\mathcal{O})$$

For example, if A is a graded algebra and $M \in \mathsf{Gr}\, A$, then $B(\mathsf{Gr}\, A, M, (1)) = \underline{\mathrm{End}}_A(M)$ as graded algebras.

For the rest of this section, we recall basic notations and properties for noncommutative invariant theory. Let A be a graded algebra and let $\sigma \in \mathrm{GrAut}\, A$, $M, N \in \mathsf{Gr}\, A$. A k-linear graded map $f : M \to N$ is called σ-linear if $f : M \to N_\sigma$ is a graded A-module homomorphism. If A is AS-Gorenstein, then by [14, Lemma 2.2], $\sigma : A \to A$ induces a σ-linear map $\underline{\mathrm{H}}^d_{\mathfrak{m}}(\sigma) : \underline{\mathrm{H}}^d_{\mathfrak{m}}(A) \to \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)$. Moreover, there exists a constant $c \in k^\times$ such that $\underline{\mathrm{H}}^d_{\mathfrak{m}}(\sigma) : \underline{\mathrm{H}}^d_{\mathfrak{m}}(A) \to \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)$ is equal to $c(\sigma^{-1})^* : A^*(\ell) \to A^*(\ell)$. The constant c^{-1} is called the homological determinant of σ, and we denote $\mathrm{hdet}\, \sigma = c^{-1}$ (see [14, Definition 2.3]). By [14, Lemma 2.5], hdet defines a group homomorphism $\mathrm{GrAut}\, A \to k^\times$.

Let A be a connected graded algebra. The trace of $\sigma \in \mathrm{GrAut}\, A$ is defined to be

$$\mathrm{Tr}_A(\sigma, t) = \sum_{i\in\mathbb{N}} \mathrm{tr}(\sigma|_{A_i})t^i.$$

See [10] for details. The Hilbert series of A is obtained as the trace of the identity map, that is,

$$H_A(t) = \mathrm{Tr}_A(\mathrm{id}, t) = \sum_{i\in\mathbb{N}} (\dim_k A_i)t^i.$$

Let G be a finite subgroup of $\mathrm{GrAut}\, A$. We write A^G for the fixed subalgebra of A. Then the following facts are well known.

(1) [14, Section 3] A splits as $A \cong A^G \oplus Q$ as graded A^G-A^G bimodules.

(2) [16, Corollary 1.12] If A is noetherian, then so is A^G.
(3) [16, Corollary 5.9] If A is noetherian, then A is finitely generated as a right and a left A^G-module, that is, $A^G \hookrightarrow A$ is a finite algebra homomorphism.
(4) [10, lemma 5.2] (Molien's theorem) The Hilbert series of A^G is given by

$$H_{A^G}(t) = |G|^{-1} \sum_{\sigma \in G} \mathrm{Tr}_A(\sigma, t).$$

(5) [14, Theorem 3.3] If A is AS-Gorenstein of dimension d, and G is a finite subgroup of $\mathrm{GrAut}\, A$ such that $\mathrm{hdet}\, \sigma = 1$ for all $\sigma \in G$, then A^G is AS-Gorenstein of dimension d.

3. Veronese subalgebras

For a connected graded algebra A and $r \in \mathbb{N}^+$, we define the r-th Veronese subalgebra of A by $A^{(r)} = \bigoplus_{i \in \mathbb{Z}} A_{ri}$. Note that $A^{(r)}$ is a subalgebra of A and a graded algebra, but not a graded subalgebra of A. Similarly, for $M \in \mathsf{Gr}\, A$, we define $M^{(r)} = \bigoplus_{i \in \mathbb{Z}} M_{ri} \in \mathsf{Gr}\, A^{(r)}$. It is easy to see that

$$V : \mathsf{Gr}\, A \to \mathsf{Gr}\, A^{(r)}, \quad M \mapsto M^{(r)}$$

is an exact functor.

Veronese subalgebras give some examples of noncommutative graded isolated singularities.

Example 3.1. [2, Proposition 5.10 (3)] If A is a noetherian connected graded algebra generated in degree 1, then the functor $V : \mathsf{Gr}\, A \to \mathsf{Gr}\, A^{(r)}$ induces an equivalence functor $V : \mathsf{tails}\, A \xrightarrow{\sim} \mathsf{tails}\, A^{(r)}$ for any $r \in \mathbb{N}^+$. Thus if A has finite global dimension, then $\mathsf{tails}\, A^{(r)}$ has finite global dimension, that is, $A^{(r)}$ is a graded isolated singularity.

Proposition 3.2. [18, Corollary 3.5] *Let $A = k\langle x_1, \ldots, x_d\rangle/(x_ix_j - \alpha_{ij}x_jx_i)$ be a weighted skew polynomial algebra with $\deg x_i \in \mathbb{N}^+$ and $d \geq 2$ where $\alpha_{ij} \in k$ such that $\alpha_{ij}\alpha_{ji} = \alpha_{ii} = 1$ for all $1 \leq i, j \leq d$. If $r \in \mathbb{N}^+$ such that $\gcd(\deg x_i, r) = 1$ for all $i = 1, \ldots, d$, then $V : \mathsf{Gr}\, A \to \mathsf{Gr}\, A^{(r)}$ induces an equivalence functor $V : \mathsf{tails}\, A \xrightarrow{\sim} \mathsf{tails}\, A^{(r)}$. Thus $\mathsf{tails}\, A^{(r)}$ has finite global dimension, that is, $A^{(r)}$ is a graded isolated singularity.*

Let A be a connected graded algebra, and let $r \in \mathbb{N}^+$. We define a graded algebra automorphism σ_r of A by $\sigma_r(a) = \xi^{\deg a} a$, where ξ is a primitive r-th root of unity. We write $G = \langle \sigma_r \rangle$ for the finite cyclic subgroup of $\mathrm{GrAut}(A)$ generated by σ_r. Then

$$A^G{}_i = \begin{cases} A_i & \text{for } r \mid i, \\ 0 & \text{for } r \nmid i, \end{cases}$$

so $A^G = A^{(r)}$ as an ungraded subalgebra of A, and A^G can be regarded as the r-th Veronese subalgebra with the grading as a graded subalgebra of A.

Since A^G and $A^{(r)}$ have different gradings, we see $\mathsf{gr}\, A^G \not\cong \mathsf{gr}\, A^{(r)}$, however it follows from [2, Remarks (1) p. 260] that there is an equivalence functor

$$F : \mathsf{gr}\, A^G \xrightarrow{\sim} (\mathsf{gr}\, A^{(r)})^r, \quad M \mapsto (M^{(r)}, M(1)^{(r)}, \ldots, M(r-1)^{(r)}).$$

In fact, we have an isomorphism of algebraic triples

$$(\mathsf{gr}\, A^G, M, (1)) \cong ((\mathsf{gr}\, A^{(r)})^r, (M^{(r)}, M(1)^{(r)}, ..., M(r-1)^{(r)}), s) \tag{3.1}$$

where the shift $s \in \mathrm{Aut}(\mathsf{gr}\, A^{(r)})^r$ is given by

$$s(N_0, N_1, \ldots, N_{r-1}) := (N_1, N_2, \ldots, N_0(1)).$$

Note that $M(r)^{(r)} = M^{(r)}(1)$ in $\mathsf{gr}\, A^{(r)}$.

Proposition 3.3. *Let A be a noetherian connected graded algebra, and let $G = \langle \sigma_r \rangle$ be the cyclic subgroup generated by σ_r. Then* $\operatorname{gldim} \mathsf{tails}\, A^{(r)} = \operatorname{gldim} \mathsf{tails}\, A^G$. *In particular, $A^{(r)}$ is a graded isolated singularity if and only if so is A^G.*

Proof. Since the equivalence functor $F : \mathsf{gr}\, A^G \to (\mathsf{gr}\, A^{(r)})^r$ induces an equivalence functor $F : \mathsf{tails}\, A^G \to (\mathsf{tails}\, A^{(r)})^r$, the assertion follows. □

Note that if A is AS-Gorenstein and $r \in \mathbb{N}^+$ such that $r \mid \ell$, then $\operatorname{hdet} \sigma_r = 1$, so A^G is AS-Gorenstein (see [14, Theorem 3.6]).

4. Serre functors

Definition 4.1. Let C be a k-linear category such that $\dim_k \operatorname{Hom}_{\mathsf{C}}(\mathcal{M}, \mathcal{N}) < \infty$ for all $\mathcal{M}, \mathcal{N} \in \mathsf{C}$. An autoequivalence $S : \mathsf{C} \to \mathsf{C}$ is called the Serre functor for C if we have a functorial isomorphism

$$\operatorname{Hom}_{\mathsf{C}}(\mathcal{M}, \mathcal{N}) \cong \operatorname{Hom}_{\mathsf{C}}(\mathcal{N}, S(\mathcal{M}))^*$$

for all $\mathcal{M}, \mathcal{N} \in \mathsf{C}$.

Note that the Serre functor is unique if it exists. Let A be an AS-Gorenstein isolated singularity. Since $\omega_A \cong A_\nu(-\ell)$, the autoequivalence $- \otimes_A \omega_A \cong (-)_\nu(-\ell) : \mathsf{gr}\, A \to \mathsf{gr}\, A$ induces an autoequivalence $- \otimes^{\mathrm{L}}_{\mathcal{A}} \omega_{\mathcal{A}} : \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A) \to \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$. It was shown in [19] that $- \otimes^{\mathrm{L}}_{\mathcal{A}} \omega_{\mathcal{A}}[d-1]$ is the Serre functor for $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$, so we have

$$\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \cong \underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{N}, \mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{A}} \omega_{\mathcal{A}}[d-1])^* \cong \underline{\operatorname{Ext}}^{d-1-i}_{\mathcal{A}}(\mathcal{N}, \mathcal{M} \otimes_{\mathcal{A}} \omega_{\mathcal{A}})^* \tag{4.1}$$

for all $i \in \mathbb{Z}$.

Let A be a balanced Cohen-Macaulay algebra of depth d. For any $M \in \mathsf{gr}\, A$ and any $i \geq d$, $\Omega^i M$ is maximal Cohen-Macaulay. If M is maximal Cohen-Macaulay, then so is ΩM. We denote by $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ the stable category of $\mathsf{CM}^{\mathsf{gr}}(A)$. Thus $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the same objects as $\mathsf{CM}^{\mathsf{gr}}(A)$ and the morphism set is given by

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) = \operatorname{Hom}_{\mathsf{Gr}\, A}(M, N)/P(M, N)$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$, where $P(M, N)$ consists of the degree zero A-module homomorphisms that factor through a projective module in $\mathsf{Gr}\, A$. The syzygy gives a functor $\Omega : \underline{\mathsf{CM}}^{\mathsf{gr}}(A) \to \underline{\mathsf{CM}}^{\mathsf{gr}}(A)$.

Let A be an AS-Gorenstein algebra. Then maximal Cohen-Macaulay modules coincide with totally reflexive modules. So for $M \in \mathsf{CM}^{\mathsf{gr}}(A)$, by taking a free resolution of $M^\vee \in \mathsf{CM}^{\mathsf{gr}}(A^{\mathrm{op}})$ and applying $(-)^\vee$, we have the following long exact sequence

$$0 \to M \to G_0 \xrightarrow{d_{-1}} G_1 \to \cdots \to G_{n-1} \xrightarrow{d_{-n}} G_n \to \cdots,$$

where each G_i is a graded free A-module. For any $n \in \mathbb{N}$, we can define $\Omega^{-n} M$ by $\operatorname{Im} d_{-n}$. Moreover, note that we have

$$\underline{\operatorname{Ext}}^i_A(M, N) \cong \underline{\operatorname{Ext}}^{i+j}_A(M, \Omega^j N) \tag{4.2}$$

for any $M \in \mathsf{CM}^{\mathsf{gr}}(A)$, $N \in \mathsf{gr}\, A$ and any $i, j > 0$.

Theorem 4.2. [5] *If A is AS-Gorenstein, then $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ is a triangulated category with respect to the translation functor $M[-1] = \Omega M$. If M, N are maximal Cohen-Macaulay modules, then*

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N[i]) \cong \operatorname{Ext}^i_{\mathsf{Gr}\, A}(M, N)$$

for all $i \geq 1$.

The following theorem plays an essential role in the proof of our first main result.

Theorem 4.3. *Let A be an AS-Gorenstein algebra of dimension $d \geq 2$, and assume that $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A)$ has the Serre functor $- \otimes^{\mathrm{L}}_{\mathcal{A}} \omega_{\mathcal{A}}[d-1]$, then there exists a functorial isomorphism*

$$\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) \cong \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(N, M \otimes_A \omega_A[d-1])^*,$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$, that is, $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the Serre functor $- \otimes_A \omega_A[d-1]$.

Lemma 4.4. *Keep the hypotheses of Theorem 4.3. Then there exists a functorial isomorphism*

$$\underline{\mathrm{Hom}}_A(M, N) \cong \lim_{n\to\infty} \underline{\mathrm{Ext}}^{d-1}_A(N_{\geq n}, M \otimes_A \omega_A)^*.$$

for any $M \in \mathsf{gr}\, A$ and $N \in \mathsf{CM}^{\mathsf{gr}}(A)$.

Proof. Take an exact sequence

$$0 \to M_{\geq n} \to M \to M/M_{\geq n} \to 0\ ,$$

and applying $\underline{\mathrm{Hom}}_A(-, N)$, we have an exact sequence

$$\underline{\mathrm{Hom}}_A(M/M_{\geq n}, N) \to \underline{\mathrm{Hom}}_A(M, N) \to \underline{\mathrm{Hom}}_A(M_{\geq n}, N) \to \underline{\mathrm{Ext}}^1_A(M/M_{\geq n}, N)\ .$$

Since $M/M_{\geq n}$ is finite dimensional over k and $\mathrm{depth}_A N = d \geq 2$, both the first term and the fourth term in the exact sequence vanish. Consequently, we have the desired isomorphism by

$$\begin{aligned}\underline{\mathrm{Hom}}_A(M, N) &\cong \lim_{n\to\infty} \underline{\mathrm{Hom}}_A(M_{\geq n}, N) \overset{(2.1)}{\cong} \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \\ &\overset{(4.1)}{\cong} \underline{\mathrm{Ext}}^{d-1}_{\mathcal{A}}(\mathcal{N}, \mathcal{M} \otimes_{\mathcal{A}} \omega_{\mathcal{A}})^* \overset{(2.1)}{\cong} \lim_{n\to\infty} \underline{\mathrm{Ext}}^{d-1}_A(N_{\geq n}, M \otimes_A \omega_A)^*.\end{aligned}$$

□

Proof of Theorem 4.3. Let $0 \to K \to F \xrightarrow{\varepsilon} N \to 0$ be an exact sequence where F is a free A-module. Then

$$\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) \cong \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, K[1]) \overset{\text{Thm 4.2}}{\cong} \mathrm{Ext}^1_{\mathsf{Gr}\, A}(M, K). \tag{4.3}$$

Moreover, applying $\underline{\mathrm{Hom}}_A(-, M \otimes_A \omega_A)$ to the exact sequence

$$0 \to K_{\geq n} \to F_{\geq n} \to N_{\geq n} \to 0$$

yields an exact sequence

$$\mathbb{E}^{d-2}(F_{\geq n}) \to \mathbb{E}^{d-2}(K_{\geq n}) \to \mathbb{E}^{d-1}(N_{\geq n}) \to \mathbb{E}^{d-1}(F_{\geq n}) \to \mathbb{E}^{d-1}(K_{\geq n})\ ,$$

here we put $\mathbb{E}^i(X) = \underline{\mathrm{Ext}}^i_A(X, M \otimes_A \omega_A)$ for short. Applying $(-)^*$ to this sequence, we obtain an exact sequence

$$\mathbb{E}^{d-1}(K_{\geq n})^* \to \mathbb{E}^{d-1}(F_{\geq n})^* \to \mathbb{E}^{d-1}(N_{\geq n})^* \to \mathbb{E}^{d-2}(K_{\geq n})^* \to \mathbb{E}^{d-2}(F_{\geq n})^*\ .$$

Since $M \otimes_A \omega_A$ is maximal Cohen-Macaulay, there exist isomorphisms

$$\mathbb{E}^{d-2}(F_{\geq n}) = \underline{\mathrm{Ext}}^{d-2}_A(F_{\geq n}, M \otimes_A \omega_A) \cong \underline{\mathrm{Ext}}^{d-1}_A(F/F_{\geq n}, M \otimes_A \omega_A) = 0 \text{ and} \tag{4.4}$$

$$\mathbb{E}^{d-2}(K_{\geq n}) = \underline{\mathrm{Ext}}^{d-2}_A(K_{\geq n}, M \otimes_A \omega_A) \cong \underline{\mathrm{Ext}}^{d-2}_A(K, M \otimes_A \omega_A), \tag{4.5}$$

so by (4.4), the fifth term in the above exact sequence vanishes. Consequently, by Lemma 4.4, we have a commutative diagram

$$\begin{array}{ccccccccc}
\underline{\mathrm{Hom}}_A(M,K) & \longrightarrow & \underline{\mathrm{Hom}}_A(M,F) & \longrightarrow & \underline{\mathrm{Hom}}_A(M,N) & \longrightarrow & \underline{\mathrm{Ext}}^1_A(M,K) & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \vdots & & \\
\lim\limits_{n\to\infty}\mathbb{E}^{d-1}(K_{\geq n})^* & \longrightarrow & \lim\limits_{n\to\infty}\mathbb{E}^{d-1}(F_{\geq n})^* & \longrightarrow & \lim\limits_{n\to\infty}\mathbb{E}^{d-1}(N_{\geq n})^* & \longrightarrow & \lim\limits_{n\to\infty}\mathbb{E}^{d-2}(K_{\geq n})^* & \longrightarrow & 0,
\end{array}$$

so we get

$$\underline{\mathrm{Ext}}^1_A(M,K) \cong \lim_{n\to\infty}\underline{\mathrm{Ext}}^{d-2}_A(K_{\geq n}, M\otimes_A\omega_A)^* \overset{(4.5)}{\cong} \underline{\mathrm{Ext}}^{d-2}_A(K, M\otimes_A\omega_A)^*. \tag{4.6}$$

Hence we have the desired assertion by

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M,N) &\overset{(4.3)}{\cong} \mathrm{Ext}^1_{\mathsf{Gr}\,A}(M,K) \overset{(4.6)}{\cong} \mathrm{Ext}^{d-2}_{\mathsf{Gr}\,A}(K, M\otimes_A\omega_A)^* \\
&\cong \mathrm{Ext}^{d-1}_{\mathsf{Gr}\,A}(N, M\otimes_A\omega_A)^* \overset{\text{Thm 4.2}}{\cong} \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(N, M\otimes_A\omega_A[d-1])^*.
\end{aligned}$$

□

We have the first main result of this paper.

Corollary 4.5. *Let A be an AS-Gorenstein algebra of dimension $d \geq 2$. Then the following are equivalent.*

1. *A is a graded isolated singularity.*
2. *$\mathsf{D^b}(\mathsf{tails}\,A)$ has the Serre functor $-\otimes^{\mathsf{L}}_{\mathcal{A}}\omega_{\mathcal{A}}[d-1]$.*
3. *$\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the Serre functor $-\otimes_A\omega_A[d-1]$.*

Proof. (1) ⇒ (2) follows from [19, Appendix A] and (2) ⇒ (3) was already proved in Theorem 4.3, so we now prove (3) ⇒ (1). For all $M, N \in \mathsf{gr}\,A$ and all $i > 0$, we can compute

$$\begin{aligned}
\mathrm{Ext}^{d+i}_{\mathsf{tails}\,A}(\mathcal{M},\mathcal{N}) &\cong \lim_{n\to\infty}\mathrm{Ext}^{d+i}_{\mathsf{Gr}\,A}(M_{\geq n}, N) \\
&\cong \lim_{n\to\infty}\mathrm{Ext}^{i}_{\mathsf{Gr}\,A}(\Omega^d(M_{\geq n}), N) \\
&\overset{(a)}{\cong} \lim_{n\to\infty}\mathrm{Ext}^{d+i}_{\mathsf{Gr}\,A}(\Omega^d(M_{\geq n}), \Omega^d N) \\
&\cong \lim_{n\to\infty}\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(\Omega^d(M_{\geq n}), \Omega^d N[d+i]) \\
&\overset{(b)}{\cong} \lim_{n\to\infty}\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(\Omega^d N[d+i], \Omega^d(M_{\geq n})\otimes_A\omega_A[d-1])^* \\
&\cong \lim_{n\to\infty}\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(\Omega^d N[i+1], (\Omega^d(M_{\geq n}))_\nu(-\ell))^* \\
&\cong \lim_{n\to\infty}\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(\Omega^d N[i+1], \Omega^d(M_\nu(-\ell)_{\geq n+\ell}))^* \qquad (4.7)
\end{aligned}$$

where (a) is by (4.2) and (b) is by the Serre functor for $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$. Since the degree i parts of $\Omega^d(M_\nu(-\ell)_{\geq n+\ell})$ are zero for all $i < n+\ell$, and $n+\ell$ is greater than the degrees of minimal generators for $\Omega^{-i-1}(\Omega^d N)$ for $n \gg 0$, we have

$$\lim_{n\to\infty}\mathrm{Hom}_{\mathsf{gr}\,A}(\Omega^{-i-1}(\Omega^d N), \Omega^d(M_\nu(-\ell)_{\geq n+\ell})) = 0.$$

Hence (4.7) is zero. □

As an application, we have the Auslander-Reiten duality. We define the Auslander-Reiten translation

$$\tau : \underline{\mathsf{CM}}^{\mathsf{gr}}(A) \xrightarrow{\Omega^d_{A^{\mathrm{op}}}\operatorname{Tr}(-)} \underline{\mathsf{CM}}^{\mathsf{gr}}(A^{\mathrm{op}}) \xrightarrow{\operatorname{Hom}_{A^{\mathrm{op}}}(-,\omega_A)} \underline{\mathsf{CM}}^{\mathsf{gr}}(A).$$

Corollary 4.6. *Let A be an AS-Gorenstein algebra of dimension $d \geq 2$. Then the following are equivalent.*

(1) *A is a graded isolated singularity.*
(2) (Auslander-Reiten duality) *There exists a functorial isomorphism*

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M,N)^* \cong \operatorname{Ext}^1_{\mathsf{Gr}\,A}(N,\tau M)$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$.

Proof. Similar to [9, Proposition 3.4], we can show that $\tau \cong -\otimes_A \omega_A[d-2]$ as functors $\underline{\mathsf{CM}}^{\mathsf{gr}}(A) \to \underline{\mathsf{CM}}^{\mathsf{gr}}(A)$. The result follows from combining this fact, Corollary 4.5 and Theorem 4.2. □

For the rest of this section, we give concrete examples.

Example 4.7. Let

$$A = k\langle x,y\rangle/(\alpha xy^2 + \beta yxy + \alpha y^2x + \gamma x^3, \alpha yx^2 + \beta xyx + \alpha x^2y + \gamma y^3), \quad \deg x = \deg y = 1$$

where $\alpha, \beta, \gamma \in k$ are generic scalars. By [1], A is a cubic AS-regular algebra of type A with $\operatorname{gldim} A = 3$ and Gorenstein parameter 4. Let $G = \langle \sigma_4 \rangle$ be the subgroup of $\operatorname{GrAut}(A)$ generated by σ_4. Then A^G is AS-Gorenstein of dimension 3. Since $A^{(4)}$ is a graded isolated singularity (see Example 3.1), so is A^G by Proposition 3.3. It follows that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$ has the Serre functor.

Example 4.8. Let

$$A = k\langle x,y,z\rangle/(xy - \alpha yx, yz - \beta zy, zx - \gamma xz), \quad \deg x = 1, \deg y = 2, \deg z = 4$$

be a weighted skew polynomial algebra where α, β, γ are nonzero scalars. One can check that A is a 3-dimensional AS-regular algebra of Gorenstein parameter 7. Let $G = \langle \sigma_7 \rangle$ be the subgroup of $\operatorname{GrAut}(A)$ generated by σ_7. Then A^G is AS-Gorenstein of dimension 3. Since $A^{(7)}$ is a graded isolated singularity by Proposition 3.2, so is A^G by Proposition 3.3. It follows that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$ has the Serre functor.

We can give another example by using [20, Proposition 5.2].

Example 4.9. Let

$$S = k\langle x,y,z\rangle/(xy + yx + z^2, xz + zx + y^2, yz + zy), \quad \deg x = \deg y = \deg z = 1.$$

It is easy to check that S is a 3-dimensional Koszul AS-regular algebra with a central regular element $x^2 \in S_2$. Let

$$A = S/(x^2).$$

Then $A^! \cong k[x,y,z]/(yx - z^2, xz - y^2)$. Take $w = x^2 \in A^!{}_2$, then w is a central regular element such that $S^! \cong A^!/(w)$. Moreover, we can show that the algebra $C(A) := A^![w^{-1}]_0$ is isomorphic to $k[X]/(X^4 - X)$. (Note that both algebras are 4-dimensional over k. $C(A)$ has a basis $\{1, yxw^{-1}, xzw^{-1}, zyw^{-1}\}$, and $k[X]/(X^4 - X)$ has a basis $\{1, X, X^2, X^3\}$. The assignment $yxw^{-1} \mapsto X, xzw^{-1} \mapsto X^2, zyw^{-1} \mapsto X^3$ gives an isomorphism.) Since k is algebraically closed, $k[X]/(X^4 - X)$ is a semisimple ring, so [20, Proposition 5.2] shows that $\mathsf{tails}\,A$ has finite global dimension. Because A is AS-Gorenstein, $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the Serre functor.

5. n-cluster tilting modules

5.1. **representation-finite.** Let A be a noetherian connected graded algebra. We call A representation-finite if there exist finitely many indecomposable graded maximal Cohen-Macaulay modules $X_1, \dots, X_n$ so that, up to isomorphism, the indecomposable graded maximal Cohen-Macaulay modules in $\mathsf{gr}\, A$ are precisely the degree shifts $X_i(s)$ for $1 \le i \le n$ and $s \in \mathbb{Z}$.

Proposition 5.1. [13, Lemma 2.3, Theorem 2.5] *Let A be a balanced Cohen-Macaulay algebra. If A is representation-finite, then $\underline{\mathrm{Ext}}^1_A(M, N)$ are finite dimensional over k for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$. In particular, if A is FBN and representation-finite, then A is a graded isolated singularity.*

The following is a noncommutative graded version of a classical result due to Auslander [3]. The proof in [13, Section 3] works well in our setting.

Proposition 5.2. *Let A be an AS-regular algebra of dimension 2, and let G be a finite subgroup of* $\mathrm{GrAut}\, A$ *such that* $\mathrm{hdet}\, \sigma = 1$ *for all $\sigma \in G$. Then A^G is representation-finite. In fact, the indecomposable maximal Cohen-Macaulay modules over A^G are precisely the indecomposable summands of $A(s)$.*

Unfortunately, the author can not check whether A^G in the above result is FBN or not. However, we can show that A^G is a graded isolated singularity. The proof below is inspired by the method of [9, Theorem 3.2], but we need to modify some part of the proof due to the noncommutativity.

Corollary 5.3. *Keep the hypotheses of Proposition 5.2. Then A^G is a graded isolated singularity.*

Proof. For any $X, Y \in \mathsf{CM}^{\mathsf{gr}}(A^G)$, similar to [9, Lemma 3.3], we can show

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(X, Y(i)) \cong \mathrm{Tor}_1^{A^G}(Y, \mathrm{Tr}\, X).$$

Since

$$\begin{aligned}\mathrm{Tor}_i^{A^G}(Y, \mathrm{Tr}\, X) &\cong \mathrm{Tor}_1^{A^G}(\Omega_{A^G}^{-i+1} Y, \mathrm{Tr}\, X) \cong \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(X, \Omega_{A^G}^{-i+1} Y(j)) \\ &\cong \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(X, \Omega_A^{-i+2} Y(j)[1]) \cong \underline{\mathrm{Ext}}^1_{A^G}(X, \Omega_{A^G}^{-i+2} Y)\end{aligned}$$

for any $i \ge 1$, $\mathrm{Tor}_i^{A^G}(Y, \mathrm{Tr}\, X)$ is finite dimensional by Lemma 5.1. Because A is a graded A-A^G bimodule, we have

$$\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(A \otimes^{\mathrm{L}}_{A^G} \mathrm{Tr}\, X, \omega_A) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^{G\,\mathrm{op}}}(\mathrm{Tr}\, X, \omega_A),$$

so we have the following spectral sequence

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p_{A^{\mathrm{op}}}(\mathrm{Tor}_q^{A^G}(A, \mathrm{Tr}\, X), \omega_A) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{A^{G\,\mathrm{op}}}(\mathrm{Tr}\, X, \omega_A).$$

Thus, along with [9, Theorem 3.2], we can get

$$\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(X, A)^* \cong \mathrm{Ext}^1_{\mathsf{Gr}\, A^G}(A, \tau X) \cong \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(A, \tau X[1]). \tag{5.1}$$

Because we have

$$\mathrm{Ext}^{2+i}_{\mathsf{tails}\, A}(\mathcal{M}, \mathcal{N}) \cong \lim_{n \to \infty} \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(\Omega^2_{A^G}(M_{\ge n}), \Omega^2_{A^G}(N)[2+i])$$

and every maximal Cohen-Macaulay module over A^G is a direct summand of $A(s_1)\oplus\cdots\oplus A(s_n)$, it is enough to show

$$\lim_{n\to\infty}\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(\Omega^2_{A^G}(M_{\geq n}), A(s)) = 0.$$

But, similar to Corollary 4.5, this follows from (5.1). □

Example 5.4. Let

$$A = k\langle x,y\rangle/(xy+yx), \quad \deg x = \deg y = 1.$$

We define a graded algebra automorphism $\sigma \in \operatorname{GrAut} A$ by $\sigma(x) = y, \sigma(y) = x$. By using [14, Lemma 2.6, Proposition 4.2], one can check $\operatorname{hdet}\sigma = 1$. Let $G = \langle\sigma\rangle \leq \operatorname{GrAut} A$. Then A^G is AS-Gorenstein of dimension 2 and

$$H_{A^G}(t) = \frac{1-t+t^2}{(1-t)^2(1+t^2)}$$

by Molien's theorem. It follows from Proposition 5.2 and Proposition 5.3 that A^G is representation-finite and a graded isolated singularity. But A^G is not AS-regular because $H_{A^G}(t)^{-1} \notin \mathbb{Z}[t]$. Corollary 4.5 shows that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$ has the Serre functor.

Example 5.5. Let

$$A = k\langle x,y\rangle/(xy-\alpha yx) \quad 0\neq\alpha\in k, \quad \deg x = \deg y = 1.$$

We define a graded algebra automorphism $\sigma\in\operatorname{GrAut} A$ by $\sigma(x) = \xi x, \sigma(y) = \xi^2 y$ where ξ is a primitive 3-rd root of unity. One can also check $\operatorname{hdet}\sigma = 1$. Let $G = \langle\sigma\rangle \leq \operatorname{GrAut} A$. Then A^G is AS-Gorenstein of dimension 2 and

$$H_{A^G}(t) = \frac{1-t+t^2}{(1-t)^2(1+t+t^2)}$$

Similar to Example 5.4, A^G is a graded isolated singularity, although A^G has infinite global dimension. Corollary 4.5 shows that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$ has the Serre functor.

5.2. **n-cluster tilting modules.** The notion of n-cluster tilting subcategories plays an important role from the viewpoint of higher analogue of Auslander-Reiten theory [6], [7]. It can be regarded as a natural generalization of the classical notion of representation-finiteness.

Definition 5.6. Let A be a balanced Cohen-Macaulay algebra. A graded maximal Cohen-Macaulay module $X\in\mathsf{CM}^{\mathsf{gr}}(A)$ is called an n-cluster tilting module if

$$\begin{aligned}\operatorname{add}_A\{X(s)\mid s\in\mathbb{Z}\} &= \{M\in\mathsf{CM}^{\mathsf{gr}}(A)\mid \underline{\operatorname{Ext}}^i_A(M,X) = 0\ (0<i<n)\}\\ &= \{M\in\mathsf{CM}^{\mathsf{gr}}(A)\mid \underline{\operatorname{Ext}}^i_A(X,M) = 0\ (0<i<n)\}.\end{aligned}$$

Note that A is representation-finite if and only if A has a 1-cluster tilting module. Thus if A and A^G are as in Proposition 5.2, then A^G has a 1-cluster tilting module $A\in\mathsf{CM}^{\mathsf{gr}}(A^G)$.

Lemma 5.7. *Let A be an AS-Gorenstein algebra. If A is a graded isolated singularity, then $\underline{\operatorname{Ext}}^i_A(M,N)$ are finite dimensional over k for all $i\geq 1$ and any $M\in\mathsf{CM}^{\mathsf{gr}}(A), N\in\operatorname{gr} A$.*

Proof. Put $d = \operatorname{gldim}\mathsf{tails}\, A$. By (4.2), we have $\underline{\operatorname{Ext}}^i_A(M,N)\cong\underline{\operatorname{Ext}}^{d+i}_A(M,\Omega^d_A N)$ for any $i\geq 1$. Moreover, [20, Proposition 4.3] says that $\dim_k\underline{\operatorname{Ext}}^{d+i}_A(M,\Omega^d_A N) < \infty$, so $\dim_k\underline{\operatorname{Ext}}^i_A(M,N)<\infty$. □

Lemma 5.8. *If A is a d-dimensional AS-Gorenstein algebra with the generalized Nakayama automorphism $\nu\in\operatorname{GrAut} A$ and G is a finite subgroup of $\operatorname{GrAut} A$ such that $\operatorname{hdet}\sigma = 1$ for all $\sigma\in G$, then the generalized Nakayama automorphism of A^G is given by $\nu|_{A^G}$.*

Proof. By [22, Remark 4.12], ν is in the center of $\operatorname{GrAut} A$, so $\nu|_{A^G}$ is in $\operatorname{GrAut} A^G$. Since $A \cong A^G \oplus Q$ as graded A^G-A^G bimodule, we have a composition of homomorphisms

$$\underline{\mathrm{H}}^d_{\mathfrak{m}_{A^G}}(A^G) \hookrightarrow \underline{\mathrm{H}}^d_{\mathfrak{m}_{A^G}}(A) \cong \underline{\mathrm{H}}^d_{\mathfrak{m}_A}(A) \cong A_\nu(-\ell)^* \cong A_{\nu|_{A^G}}(-\ell)^*$$

as graded A^G-A^G bimodules. Since the image of this composition is $A^G(-\ell)^*$ as a graded vector space by [14, Theorem 3.3], we have

$$\underline{\mathrm{H}}^d_{\mathfrak{m}_{A^G}}(A^G) \cong {A^G}_{\nu|_{A^G}}(-\ell)^*$$

as graded A^G-A^G bimodules. □

We now show the following noncommutative generalization of [9, Theorem 5.2] (see also [6]). The assumption (2) in the following theorem is technical, but, we will show that the assumption (2) holds for a certain class of noncommutative fixed subalgebras.

Theorem 5.9. *Let A be an AS-regular algebra of dimension $d \geq 2$, and G a finite subgroup of $\operatorname{GrAut} A$ such that $\operatorname{hdet} \sigma = 1$ for all $\sigma \in G$. If*

(1) *A^G is a graded isolated singularity, and*
(2) *$\underline{\mathrm{End}}_{A^G}(A)$ is a finitely generated graded free module over A,*

then A^G has a $(d-1)$-cluster tilting module $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$.

Proof. By [23, Lemma 4.15], we have $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$. Put $\mathsf{A} := \operatorname{add}_{A^G}\{A(s) \mid s \in \mathbb{Z}\}$. We show the following claims:

(i) $\underline{\mathrm{Ext}}^i_{A^G}(A, A) = 0$ for any $0 < i < d-1$.
(ii) $M \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ satisfying $\underline{\mathrm{Ext}}^i_{A^G}(A, M) = 0$ for any $0 < i < d-1$ belongs to A.
(iii) $M \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ satisfying $\underline{\mathrm{Ext}}^i_{A^G}(M, A) = 0$ for any $0 < i < d-1$ belongs to A.

(i): Consider the following spectral sequence

$$E_2^{p,q} = \underline{\mathrm{Ext}}^p_A(A/A_{\geq n}, \underline{\mathrm{Ext}}^q_{A^G}(A, A)) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{A^G}(A/A_{\geq n}, A) = E^{p+q}.$$

Since $\underline{\mathrm{Ext}}^q_{A^G}(A, A)$ are finite dimensional for $1 \leq q$ by Lemma 5.7, we have $E_2^{p,q} = 0$ for all $1 \leq p$ and $1 \leq q$, so there exist exact sequences

$$E^i \to E_2^{0,i} \to E_2^{i+1,0} \to E^{i+1}$$

for all $0 < i < d-1$. But it follows from $\operatorname{depth}_{A^G} A = d$ that the first term and the fourth term in the exact sequence both vanish. Moreover, $\operatorname{depth}_A \underline{\mathrm{End}}_{A^G}(A) = d$ implies the third term also vanishes. Hence we have

$$E_2^{0,i} = \underline{\mathrm{Hom}}_A(A/A_{\geq n}, \underline{\mathrm{Ext}}^i_{A^G}(A, A)) = 0$$

for any $0 < i < d-1$. However $\underline{\mathrm{Ext}}^i_{A^G}(A, A)$ is finite dimensional, so we get $\underline{\mathrm{Ext}}^i_{A^G}(A, A) = 0$.

(ii): Let $M \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ such that $\underline{\mathrm{Ext}}^i_{A^G}(A, M) = 0$ for any $0 < i < d-1$. Since $\underline{\mathrm{Hom}}_{A^G}(A, M)$ is the zeroth cohomology of $\mathrm{R}\underline{\mathrm{Hom}}_{A^G}(A, M)$, we have a distinguished triangle

$$\underline{\mathrm{Hom}}_{A^G}(A, M) \to \mathrm{R}\underline{\mathrm{Hom}}_{A^G}(A, M) \to L \to$$

where L has cohomologies concentrated in cohomological degree $\geq d-1$. This gives a distinguished triangle

$$\mathrm{R}\underline{\mathrm{Hom}}_A(k, \underline{\mathrm{Hom}}_{A^G}(A, M)) \to \mathrm{R}\underline{\mathrm{Hom}}_A(k, \mathrm{R}\underline{\mathrm{Hom}}_{A^G}(A, M)) \to \mathrm{R}\underline{\mathrm{Hom}}_A(k, L) \to \ . \quad (5.2)$$

Since

$$\mathrm{R}\underline{\mathrm{Hom}}_A(k, \mathrm{R}\underline{\mathrm{Hom}}_{A^G}(A, M)) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^G}(k \otimes^{\mathrm{L}}_A A, M) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^G}(k, M)$$

and $\operatorname{depth}_{A^G} M = d$, the second term of (5.2) has cohomologies concentrated in cohomological degree $\geq d$. Moreover, since L has cohomologies concentrated in cohomological degree $\geq d-1$, so has the third term of (5.2). Thus the long exact sequence of cohomologies of (5.2) shows $\operatorname{depth}_A \underline{\operatorname{Hom}}_{A^G}(A, M) \geq d$, and so $\operatorname{pd}_A \underline{\operatorname{Hom}}_{A^G}(A, M) = 0$ by the Auslander-Buchsbaum formula [12]. The split inclusion

$$M \cong \underline{\operatorname{Hom}}_{A^G}(A^G, M) \hookrightarrow \underline{\operatorname{Hom}}_{A^G}(A, M)$$

over A^G implies that $M \in \mathsf{A}$.

(iii): Let $M \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ such that $\underline{\operatorname{Ext}}^i_{A^G}(M, A) \cong \bigoplus_{s \in \mathbb{Z}} \operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)}(M, A(s)[i]) = 0$ for any $0 < i < d-1$. By using the Serre functor $- \otimes_{A^G} \omega_{A^G}[d-1]$ of $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$, we have $\underline{\operatorname{Ext}}^i_{A^G}(A, M \otimes_{A^G} \omega_{A^G}) = 0$ for any $0 < i < d-1$. Let $\nu \in \operatorname{GrAut} A$ be the generalized Nakayama automorphism of A. Then $\omega_{A^G} \cong {A^G}_{\nu|_{A^G}}(-\ell)$ by Lemma 5.8. Since $A_{\nu^{-1}} \cong A$ as graded right A-modules, we see $A_{\nu^{-1}|_{A^G}} \cong A_{\nu^{-1}} \cong A$ as graded right A^G-modules, so it follows that

$$\begin{aligned} \underline{\operatorname{Ext}}^i_{A^G}(A, M \otimes_{A^G} \omega_{A^G}) &\cong \underline{\operatorname{Ext}}^i_{A^G}(A, M_{\nu|_{A^G}}(-\ell)) \\ &\cong \underline{\operatorname{Ext}}^i_{A^G}(A_{\nu^{-1}|_{A^G}}, M(-\ell)) \cong \underline{\operatorname{Ext}}^i_{A^G}(A, M(-\ell)). \end{aligned}$$

Thus we obtain $\underline{\operatorname{Ext}}^i_{A^G}(A, M) = 0$ for any $0 < i < d-1$. The result follows by (ii). □

Theorem 5.10. *Let A and A^G be as in Theorem 5.9, and let $\Lambda = \underline{\operatorname{End}}_{A^G} A$. Then* $\operatorname{gldim} \Lambda = d$.

Proof. Put $\mathsf{A} := \operatorname{add}_{A^G}\{A(s) \mid s \in \mathbb{Z}\}$. Note that Λ is a graded algebra, and A has canonically a graded Λ-A^G bimodule structure. Moreover, it is well known that $\underline{\operatorname{Hom}}_{A^G}(A, -)$ induces an equivalence of categories between A and the category of finitely generated graded projective right Λ-modules.

Let $N \in \mathsf{gr}\,\Lambda$ and take a projective presentation $P_1 \to P_0 \to N \to 0$. Since we can write $P_i = \underline{\operatorname{Hom}}_{A^G}(A, X_i)$ where $X_i \in \mathsf{A}$ for each i, we have an exact sequence

$$0 \to \underline{\operatorname{Hom}}_{A^G}(A, M_1) \to \underline{\operatorname{Hom}}_{A^G}(A, X_1) \to \underline{\operatorname{Hom}}_{A^G}(A, X_0) \to N \to 0$$

such that $0 \to M_1 \to X_1 \to X_0$ is exact and $X_0, X_1 \in \mathsf{A}$.

Since $\underline{\operatorname{Hom}}_{A^G}(A, X_1)$ is a finitely generated A^G-module, so is $\underline{\operatorname{Hom}}_{A^G}(A, M_1)$, thereby we can take $f_i \in \operatorname{Hom}_{\mathsf{Gr}\,A^G}(A, M_1(-s_i))$ such that $f_1, \ldots, f_n$ generate $\underline{\operatorname{Hom}}_{A^G}(A, M_1)$ and include a set of generators of $\underline{\operatorname{Hom}}_{A^G}(A^G, M_1) \cong M_1$. We define the A^G-module homomorphism

$$\phi : \bigoplus_{i=1}^{n} A(s_i) \to M_1, \quad (a_1, \ldots, a_n) \mapsto f_1(a_1) + \cdots + f_n(a_n).$$

Clearly ϕ is surjective, and

$$\underline{\operatorname{Hom}}_{A^G}(A, \phi) : \underline{\operatorname{Hom}}_{A^G}(A, \bigoplus_{i=1}^{n} A(s_i)) \to \underline{\operatorname{Hom}}_{A^G}(A, M_1)$$

is also surjective. So we have the short exact sequences

$$0 \to M_2 \to \bigoplus_{i=1}^n A(s_i) \xrightarrow{\phi} M_1 \to 0$$

in $\mathsf{gr}\,A^G$, where $M_2 = \operatorname{Ker} \phi$, and

$$0 \to \underline{\operatorname{Hom}}_{A^G}(A, M_2) \to \bigoplus_{i=1}^n \Lambda(s_i) \xrightarrow{\underline{\operatorname{Hom}}_{A^G}(A,\phi)} \underline{\operatorname{Hom}}_{A^G}(A, M_1) \to 0$$

in $\mathsf{gr}\,\Lambda$. Repeating this step, we can make exact sequences

$$0 \to M_{d-1} \to \bigoplus_{i=1}^{n_{d-1}} A(s_{d-1,i}) \to \cdots \to \bigoplus_{i=1}^{n_2} A(s_{2,i}) \to X_1 \xrightarrow{\xi} X_0 \to \operatorname{Coker}\xi \to 0 \tag{5.3}$$

in $\mathsf{gr}\,A^G$, and

$$0 \to \underline{\operatorname{Hom}}_{A^G}(A, M_{d-1}) \to \bigoplus_{i=1}^{n_{d-1}} \Lambda(s_{d-1,i}) \to \cdots \to \bigoplus_{i=1}^{n_2} \Lambda(s_{2,i}) \to P_1 \to P_0 \to N \to 0 \tag{5.4}$$

in $\mathsf{gr}\,\Lambda$. Because X_0, X_1 and each $\bigoplus_{i=1}^{n_j} A(s_{j,i})$ in (5.3) are maximal Cohen-Macaulay modules, so is M_{d-1}. Since we have an exact sequence $0 \to M_{j+1} \to \bigoplus_{i=1}^{n_{j+1}} A(s_{j+1,i}) \to M_j \to 0$ satisfying that

$$0 \longrightarrow \underline{\operatorname{Hom}}_{A^G}(A, M_{j+1}) \longrightarrow \bigoplus_{i=1}^{n_{j+1}} \Lambda(s_{j+1,i}) \longrightarrow \underline{\operatorname{Hom}}_{A^G}(A, M_j) \longrightarrow 0$$

is exact and $\underline{\operatorname{Ext}}^t_{A^G}(A, A) = 0$ for any $0 < t < d-1$, we see $\underline{\operatorname{Ext}}^t_{A^G}(A, M_{d-1}) = 0$ for any $0 < t < d-1$. By Theorem 5.9, we have $M_{d-1} \in \mathsf{A}$ and so $\underline{\operatorname{Hom}}_{A^G}(A, M_{d-1})$ is projective over Λ. Hence $\operatorname{pd}_\Lambda N \le d$ from (5.4).

On the other hand, we consider the graded Λ-module $\underline{\operatorname{Hom}}_{A^G}(A, k)$ and a projective resolution

$$\cdots \longrightarrow \underline{\operatorname{Hom}}_{A^G}(A, X_i) \longrightarrow \cdots \longrightarrow \underline{\operatorname{Hom}}_{A^G}(A, X_0) \longrightarrow \underline{\operatorname{Hom}}_{A^G}(A, k) \longrightarrow 0 \tag{5.5}$$

where $X_i \in \mathsf{A}$ for each i. (5.5) can be viewed as a long exact sequence of graded A-modules, but we have $\operatorname{depth}_A \underline{\operatorname{Hom}}_{A^G}(A, X_i) = d$ for each i, and $\operatorname{depth}_A \underline{\operatorname{Hom}}_{A^G}(A, k) = 0$ because

$$\underline{\operatorname{Hom}}_A(k, \underline{\operatorname{Hom}}_{A^G}(A, k)) \cong \underline{\operatorname{Hom}}_{A^G}(k \otimes_A A, k)) \cong \underline{\operatorname{Hom}}_{A^G}(k, k) \neq 0,$$

so (5.5) of the length less than d would contradict to the depth lemma (cf. [4, Proposition 1.2.9]). □

5.3. **The second main result.** Let A be a connected graded algebra and G a finite subgroup of $\operatorname{GrAut} A$. Then the skew group algebra $A * G$ is an $\mathbb{N}$-graded algebra defined by $A * G = \bigoplus_{i \in \mathbb{N}} (A_i \otimes_k kG)$ as a graded vector space with the multiplication

$$(a \otimes \sigma)(a' \otimes \sigma') = a\sigma(a') \otimes \sigma\sigma'$$

for any $a, a' \in A$ and $\sigma, \sigma' \in G$.

The following theorem is an analogue of [9, Theorem 4.2] for a large class of noncommutative graded algebras and cyclic subgroups. In [18], Mori proved an ungraded version of this result. In the proof, we use the notation introduced in Section 3.

Theorem 5.11. *Let $A = k\langle x_1, \ldots, x_n\rangle/I$ be a connected graded algebra and $r \in \mathbb{N}^+$. Let $G = \langle \sigma_r \rangle$ be the finite cyclic subgroup of* $\operatorname{GrAut}(A)$ *generated by σ_r. Put $\sigma := \sigma_r$. If*

- *A is a noetherian domain satisfying χ,* $\operatorname{depth}_A A \ge 2$*,*
- *$(\mathcal{A}, (r))$ is ample for* $\mathsf{tails}\,A$ *in the sense of* [18, Definition 2.4]*, and*
- *there exists $1 \le p \le n$ such that* $\gcd(\deg x_p, r) = 1$*,*

then

$$\Phi : A * G \to \underline{\operatorname{End}}_{A^G}(A), \qquad a \otimes \sigma^s \longmapsto [b \mapsto a\sigma^s(b)]$$

is a graded algebra isomorphism.

Proof. It is easy to check that Φ is a graded algebra homomorphism, so it is enough to show

(1) $\Phi|_{(A*G)_m}$ is injective for all $m \in \mathbb{Z}$, and
(2) $H_{A*G}(t) = H_{\underline{\operatorname{End}}_{A^G}(A)}(t)$.

(1): Let $\sum_{s=0}^{r-1} a_s \otimes \sigma^s \in (A * G)_m = A_m \otimes_k kG$ such that

$$\Phi(\sum_{s=0}^{r-1} a_s \otimes \sigma^s)(b) = 0$$

for any $b \in A$. Since $\gcd(\deg x_p, r) = 1$ for some p, we see that for any $0 \le i \le r-1$, there exists $t_i \in \mathbb{N}$ such that $A_{t_i r+i} \neq 0$, so we can take $0 \neq b_i \in A_{t_i r+i}$. Then it follows that

$$0 = \Phi(\sum_{s=0}^{r-1} a_s \otimes \sigma^s)(b_i) = \sum_{s=0}^{r-1} a_s \sigma^s(b_i) = \sum_{s=0}^{r-1} a_s \xi^{s(t_i r+i)} b_i = (\sum_{s=0}^{r-1} a_s \xi^{si}) b_i.$$

Since A is a domain, we have $\sum_{s=0}^{r-1} a_s \xi^{si} = 0$ for all $0 \le i \le r-1$. These imply

$$M \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{r-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(r-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{r-1} & \xi^{2(r-1)} & \cdots & \xi^{(r-1)^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{r-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det M = \prod_{0 \le i < j \le r-1} (\xi^j - \xi^i) \neq 0$, we obtain $a_0 = \cdots = a_{r-1} = 0$ and so $\Phi|_{(A*G)_m}$ is injective.

(2): We have

$$\begin{aligned} \underline{\mathrm{End}}_{A^G}(A) &\cong B(\mathsf{gr}\, A^G, A, (1)) \\ &\cong B((\mathsf{gr}\, A^{(r)})^r, (A^{(r)}, A(1)^{(r)}, \dots, A(r-1)^{(r)}), s) \\ &\cong B((\mathsf{tails}\, A^{(r)})^r, (\mathcal{A}^{(r)}, \mathcal{A}(1)^{(r)}, \dots, \mathcal{A}(r-1)^{(r)}), s) \end{aligned} \tag{5.6}$$

where the last $\cong$ is by [18, Lemma 3.3] because

$$0 = \underline{\mathrm{Ext}}^t_{A^G}(k, A) \cong \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}^t_{\mathsf{gr}\, A^{(r)}}(k, A(j)^{(r)})$$

for $t = 0, 1$ says $\mathrm{depth}_{A^{(r)}} A(i)^{(r)} \ge 2$ for all i. By our hypothesis, [18, Theorem 3.4] shows

$$V : \mathsf{tails}\, A \xrightarrow{\sim} \mathsf{tails}\, A^{(r)}.$$

We now define an autoequivalence s' of $(\mathsf{gr}\, A)^r$ by

$$s'(M_0, M_1, \dots, M_{r-1}) := (M_1, M_2, \dots, M_0(r)).$$

Then $s' \in \mathrm{Aut}(\mathsf{gr}\, A)^r$ induces an autoequivalence $s' \in \mathrm{Aut}(\mathsf{tails}\, A)^r$ and

$$\begin{array}{ccc} (\mathsf{tails}\, A)^r & \xrightarrow[(V,\dots,V)]{\sim} & (\mathsf{tails}\, A^{(r)})^r \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ (\mathsf{tails}\, A)^r & \xrightarrow[(V,\dots,V)]{\sim} & (\mathsf{tails}\, A^{(r)})^r \end{array}$$

commutes. Thus

$$
\begin{aligned}
(5.6) &\cong B((\mathsf{tails}\,A)^r, (\mathcal{A}, \mathcal{A}(1), \dots, \mathcal{A}(r-1)), s') \\
&\cong B((\mathsf{gr}\,A)^r, (A, A(1), \dots, A(r-1)), s') \\
&\cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{(\mathsf{gr}\,A)^r}((A, A(1), \dots, A(r-1)), (A(i), A(i+1), \dots, A(i+r-1))) \\
&\cong \bigoplus_{i\in\mathbb{Z}} \bigoplus_{j=0}^{r-1} \mathrm{Hom}_{\mathsf{gr}\,A}(A(j), A(i+j)).
\end{aligned}
$$

where the second $\cong$ is by [18, Lemma 3.3] again. Since $\gcd(\deg x_p, r) = 1$, we see $|G| = r$. Hence it follows that

$$
\begin{aligned}
\dim_k \underline{\mathrm{End}}_{A^G}(A)_i = \dim_k \bigoplus_{j=0}^{r-1} \mathrm{Hom}_{\mathsf{gr}\,A}(A(j), A(i+j)) = r \dim_k A_i \\
= \dim_k(A_i \otimes_k kG) = \dim_k(A * G)_i \ (< \infty)
\end{aligned}
$$

for any i. □

We now have the second main result of this paper.

Corollary 5.12. *Let A be an AS-regular domain of dimension $d \geq 2$ generated in degree 1. Take $r \in \mathbb{N}^+$ such that $r \mid \ell$ and let $G = \langle \sigma_r \rangle$ be the finite cyclic subgroup of* $\mathrm{GrAut}(A)$ *generated by σ_r. Then*

(1) *$A * G \cong \underline{\mathrm{End}}_{A^G}(A)$ as graded algebras.*
(2) *A^G is a graded isolated singularity, and $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ is a $(d-1)$-cluster tilting module.*

Proof. In this case, $(\mathcal{A}, (r))$ is ample for $\mathsf{tails}\,A$ by Example 3.1 and [18, Theorem 3.4], so A satisfies the assumption of Theorem 5.11. Thus (1) holds. Moreover, Example 3.1 and Proposition 3.3 imply A^G is a graded isolated singularity. We see that $\underline{\mathrm{End}}_{A^G}(A) \cong A * G$ is graded free over A. Hence we obtain the result by Theorem 5.9. □

Example 5.13. Recall the setting of Example 4.7. By Corollary 5.12, A^G has a 2-cluster tilting module $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$. Moreover, we have $\mathrm{gldim}\, \underline{\mathrm{End}}_{A^G}(A) = 3$ by Theorem 5.10.

We remark that if A is an AS-regular algebra generated in degree 1, and $\underline{\mathrm{End}}_{A^G}(A) \cong A*G$ as graded algebras, then it follows from the study of skew group algebras [15, Lemma 13] that $\underline{\mathrm{End}}_{A^G}(A)$ satisfies not only global dimension d but also generalized Gorenstein condition (i.e, $\underline{\mathrm{End}}_{A^G}(A)$ is a generalized AS-regular algebra of dimension d). For instance, $\underline{\mathrm{End}}_{A^G}(A)$ in Example 5.13 is a generalized AS-regular algebra.

Corollary 5.14. *Let $A = k\langle x_1, \dots, x_d\rangle/(x_i x_j - \alpha_{ij} x_j x_i)$ be a weighted skew polynomial algebra with $\deg x_i \in \mathbb{N}^+$ and $d \geq 2$. Take $r \in \mathbb{N}^+$ such that $r \mid \ell$ and $\gcd(\deg x_i, r) = 1$ for all $i = 1, \dots, d$. Let $G = \langle \sigma_r \rangle$ be the finite cyclic subgroup of* $\mathrm{GrAut}(A)$ *generated by σ_r. Then*

(1) *$A * G \cong \underline{\mathrm{End}}_{A^G}(A)$ as graded algebras.*
(2) *A^G is a graded isolated singularity, and $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ is a $(d-1)$-cluster tilting module.*

Proof. The proof is similar to that of Corollary 5.12. □

Example 5.15. Recall the setting of Example 4.8. By Corollary 5.14, A^G has a 2-cluster tilting module $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$. Moreover, we have $\mathrm{gldim}\, \underline{\mathrm{End}}_{A^G}(A) = 3$ by Theorem 5.10.

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