GORENSTEIN DIMENSION AND AS-GORENSTEIN ALGEBRAS

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Abstract. The purpose of this paper is to connect the notion of Gorenstein dimension with AS-Gorenstein algebras. In particular, we show that a noetherian connected graded algebra having a balanced dualizing complex is AS-Gorenstein if the balanced dualizing complex has finite Gorenstein dimension. As a preparation, we generalize the Auslander-Bridger formula to the class of noncommutative noetherian connected graded algebras having balanced dualizing complexes.

1. Introduction

In the late 1960s, the Gorenstein dimension (G-dimension for short) for finitely generated modules was introduced by Auslander [3] and developed by Auslander and Bridger [4]. G-dimension is a generalization of projective dimension. Moreover, they proved the following characterization of Gorenstein rings. A commutative noetherian local ring \( R \) is Gorenstein if and only if every finitely generated module over \( R \) has finite G-dimension. They also proved that G-dimension satisfies Auslander-Buchsbaum-type formula, namely, if a finitely generated module \( M \) over a commutative noetherian local ring \( R \) has finite G-dimension, then G-dimension of \( M \) is given by \( \text{depth } R - \text{depth } M \). This formula is called the Auslander-Bridger formula.

Since then, relationships between G-dimension and Gorenstein rings have been studied deeply in commutative ring theory. For example, G-dimension for complexes with finitely generated cohomologies was studied by Yassemi [20] using reflexive complexes. The category of complexes of finite G-dimension is closely related to two important categories called the Auslander class and the Bass class, and some characterizations of Gorenstein rings were shown in terms of these categories (see [5, Chapter 3]). In another direction, Enochs and Jenda [8] defined a homological dimension called the Gorenstein projective dimension for non-finitely generated modules. They studied it when the ring is coherent or \( n \)-Gorenstein. For finitely generated modules over commutative noetherian rings, it coincides with the G-dimension. The Gorenstein homological dimensions have become an active area of research. See [5], [9] and [10] for more details.

Meanwhile, AS-Gorenstein algebras introduced by Artin and Schelter are an important class of algebras studied in noncommutative algebraic geometry (see [13], [14], [22] etc.). An AS-Gorenstein algebra is a noncommutative graded analogue of a commutative local Gorenstein ring.

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The purpose of this paper is to connect the notion of G-dimension with AS-Gorenstein algebras. In particular, we will give some characterizations of AS-Gorenstein algebras by using the finiteness of G-dimension. Indeed, we will prove the following statement.

**Theorem 1.1.** (See Theorem 5.7.) Let $A$ be a noetherian connected graded algebra having a balanced dualizing complex. Then the following are equivalent.

1. $A$ is AS-Gorenstein.
2. $\text{G-dim}_A X < \infty$ for any $X \in \mathcal{D}^b_{fg}(A)$.
3. $\text{G-dim}_A M < \infty$ for any $M \in \text{grmod} A$.
4. $\text{G-dim}_A k < \infty$.

This is a noncommutative version of [4, Theorem 4.20] and [5, Theorem 2.3.14]. The balanced dualizing complex introduced by Yekutieli [21] plays an important role in the study of homological properties of noncommutative algebras. For example, noncommutative versions of Bass theorem and the no-holes theorem can be proved by using balanced dualizing complexes (see [11], [19]). In the commutative case, Theorem 1.1 is proved without the assumption that the (balanced) dualizing complex exists. Remarkably, however, there is a noetherian noncommutative connected graded algebra which does not satisfy (4) $\Rightarrow$ (1), so the assumption that an algebra has a balanced dualizing complex is a necessary condition.

Let $A$ be a noetherian connected graded algebra having a balanced dualizing complex. In [6], Dong and Wu proved the following theorem. If the balanced dualizing complex of $A$ has finite projective dimension, then $A$ is AS-Gorenstein. Since G-dimension is a generalization of projective dimension, the following is a natural question to ask. If the balanced dualizing complex of $A$ has finite G-dimension, then is $A$ AS-Gorenstein? The main result of this paper is to show that the above question is true.

**Theorem 1.2.** (See Theorem 5.10.) Let $A$ be a noetherian connected graded algebra having a balanced dualizing complex $D$. Then the following are equivalent.

1. $A$ is AS-Gorenstein.
2. $\text{G-dim}_A D < \infty$.

This result can be viewed as a noncommutative version of [5, Theorem 3.3.5]. However, our proof is different from that of [5, Theorem 3.3.5] in the commutative case. [5] uses the fact that any module over a commutative ring $R$ is automatically an $R$-$R$ bimodule.

As a corollary, we see that if $A$ admits a totally reflexive module having finite injective dimension, then $A$ is AS-Gorenstein. To prove the main result, we generalize the Auslander-Bridger formula to the class of noncommutative noetherian connected graded algebras having balanced dualizing complexes (see Theorem 4.3).

2. Notations and Preliminaries

Throughout this paper, we fix a field $k$. Let $A$ be a connected graded $k$-algebra, that is, $A = \bigoplus_{i \in \mathbb{N}} A_i$, such that $A_0 = k$. We write $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$ for the unique maximal homogeneous two-sided ideal of $A$, and we view $k = A/\mathfrak{m}$ as a graded $A$-module. We denote by $\text{GrMod} A$ the category of graded right $A$-modules, and
by grmod\textsubscript{$A$} the full subcategory consisting of finitely generated graded right \textit{A}-modules. Morphisms in GrMod\textsubscript{$A$} are right \textit{A}-module homomorphisms preserving degrees.

For a graded module $M \in$ GrMod\textsubscript{$A$} and an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in$ GrMod\textsubscript{$A$} and the shift $M(n) \in$ GrMod\textsubscript{$A$} by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. For $M, N \in$ GrMod\textsubscript{$A$}, we write

$$\text{Ext}^i(A, N) = \bigoplus_{n \in \mathbb{Z}} \text{Ext}^i_{\text{GrMod}\textsubscript{$A$}}(M, N(n)).$$

Let $\tau \in \text{Aut}_k \textit{A}$ be a graded algebra automorphism. For a graded right \textit{A}-module $M \in$ GrMod\textsubscript{$A$}, we define a new graded right \textit{A}-module $M_{\tau} \in$ GrMod\textsubscript{$A$} by $M_{\tau} = M$ as graded vector spaces with the new right action $m \ast a = m\tau(a)$ for $m \in M$ and $a \in \textit{A}$.

Let $\textit{A}, \textit{B}$ be connected graded algebras. The category of graded left \textit{A}-modules is denoted by GrMod\textsubscript{$\textit{A}$op}, where $\textit{A}$op is the opposite algebra. The category of graded \textit{B}-\textit{A} bimodules is denoted by GrMod($\textit{B}$op $\otimes$ $\textit{A}$). In particular, the category of graded \textit{A}-\textit{A} bimodules is denoted by GrMod $\textit{A}$, where $\textit{A}$op = $\textit{A}$op $\otimes$ $\textit{A}$. For any $M \in$ GrMod($\textit{B}$op $\otimes$ $\textit{A}$), we denote by $M^* := \text{Hom}_k(M, k)$ the Matlis dual of $M$. By definition, $M^*$ has a graded $\textit{A}$-$\textit{B}$ bimodule structure.

The derived category of GrMod\textsubscript{$\textit{A}$} is denoted by $\mathcal{D}(\textit{A})$. We write for $\mathcal{D}_{fg}(\textit{A})$ the full subcategory of $\mathcal{D}(\textit{A})$ consisting of complexes whose cohomologies are all finitely generated \textit{A}-modules. For any $X \in \mathcal{D}(\textit{A})$, we denote by $h^i(X)$ the $i$-th cohomology module of $X$, and define

$$\sup X = \sup \{i \mid h^i(X) \neq 0\}$$

and

$$\inf X = \inf \{i \mid h^i(X) \neq 0\}.$$
Let $X \in \mathcal{D}^-(S^{\text{op}} \otimes A), Y \in \mathcal{D}^-(A^{\text{op}} \otimes B), Z \in \mathcal{D}^+(R^{\text{op}} \otimes B)$. Then there is a natural isomorphism in $\mathcal{D}^+(R^{\text{op}} \otimes S)$,

$$\mathcal{RHom}_B(X \otimes_A^L Y, Z) \cong \mathcal{RHom}_A(X, \mathcal{RHom}_B(Y, Z)).$$

(3) Let $X \in \mathcal{D}^-_f(A^{\text{op}}), Y \in \mathcal{D}^b(A^{\text{op}} \otimes B), Z \in \mathcal{D}^+(R^{\text{op}} \otimes B)$. If either $\text{pd}_{A^{\text{op}}} X < \infty$ or $\text{id}_B Z < \infty$, then there is a natural isomorphism in $\mathcal{D}(R^{\text{op}})$,

$$\mathcal{RHom}_B(Y, Z) \otimes_A^L X \cong \mathcal{RHom}_B(\mathcal{RHom}_{A^{\text{op}}}(X, Y), Z).$$

**Lemma 2.3.** (cf. [15, Lemma 1.3]) If $X \in \mathcal{D}^-(B^{\text{op}} \otimes A), Y \in \mathcal{D}^+(C^{\text{op}} \otimes A)$, then

$$\inf \mathcal{RHom}_A(X, Y) \geq \inf Y - \sup X.$$

**Lemma 2.4.** (cf. [13, Lemma 1.8]) If $X \in \mathcal{D}^-(B^{\text{op}} \otimes A), Y \in \mathcal{D}^-(A^{\text{op}} \otimes C)$, then

$$\sup X \otimes_A^L Y \leq \sup X + \sup Y.$$

Moreover, if $h_{\sup}^X X, h_{\sup}^Y Y$ are left bounded, then $\sup X \otimes_A^L Y = \sup X + \sup Y$.

### 3. Gorenstein Dimension

First, we recall the definition of G-dimension of finitely generated modules and of bounded complexes with finitely generated cohomologies. Throughout this section, $A$ is a noetherian connected graded algebra.

If $M \in \text{GrMod} A$, then we define $M^\vee = \text{Hom}_A(M, A) \in \text{GrMod} A^{\text{op}}$. Similarly, if $N \in \text{GrMod} A^{\text{op}}$, then we define $N^\vee = \text{Hom}_{A^{\text{op}}}(N, A) \in \text{GrMod} A$.

**Definition 3.1.** (cf. [4, Chapter 3], [5, Definition 1.2.3]) Let $A$ be a noetherian connected graded algebra.

(1) We say that $M \in \text{grmod} A$ is totally reflexive if

(a) The natural homomorphism $M \to M^{\vee \vee}$ is an isomorphism.

(b) $\text{Ext}_A^i(M, A) = \text{Ext}_{A^{\text{op}}}^i(M^\vee, A) = 0$ for all $i > 0$.

(2) Let $M \in \text{grmod} A$. If there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of graded right $A$-modules such that each $G_i$ is totally reflexive, then we say that $M$ has G-dimension at most $n$. If such an integer $n$ does not exist, then we say that $M$ has infinite G-dimension, and write $\text{G-dim}_A M = \infty$. If $M$ has G-dimension at most $n$ but does not have G-dimension at most $n - 1$, then we say that $M$ has G-dimension $n$, and write $\text{G-dim}_A M = n$. We set $\text{G-dim}_A 0 = -\infty$.

**Lemma 3.2.** The following hold.

(1) Let $M \in \text{grmod} A$. If $\text{G-dim}_A M < \infty$, then

$$\text{G-dim}_A M = \sup \mathcal{RHom}_A(M, A) = \sup \{i \mid \text{Ext}_A^i(M, A) \neq 0\}.$$

(2) Let $0 \to K \to G \to M \to 0$ be an exact sequence in $\text{grmod} A$. If $\text{G-dim}_A G = 0$, then $\text{G-dim}_A K = \sup \{\text{G-dim}_A M - 1, 0\}$.

**Proof.** See [5, Theorem 1.2.7] and [5, Corollary 1.2.9].

If $X \in \mathcal{D}^b(A)$, then we define $X^{\dagger} = \mathcal{RHom}_A(X, A) \in \mathcal{D}(A^{\text{op}})$. Similarly, if $Y \in \mathcal{D}^b(A^{\text{op}})$, then we define $Y^{\dagger} = \mathcal{RHom}_{A^{\text{op}}}(Y, A) \in \mathcal{D}(A)$.
**Definition 3.3.** The reflexive class $\mathcal{R}(A)$ is the full subcategory of $D^b_{fg}(A)$ consisting of the complexes $X$ satisfying

1. $X^\dagger = R\text{Hom}_A(X, A) \in D^b_{fg}(A^{\text{op}})$, and
2. The natural morphism $X \to X^{\dagger \dagger}$ is an isomorphism in $D(A)$.

**Lemma 3.4.** Let $M \in \text{grmod } A$. Then $G\text{-dim}_A M < \infty$ if and only if $M \in \mathcal{R}(A)$.

**Proof.** We prove this along the same lines as in [20, Theorem 2.7]. Assume that $G\text{-dim}_A M < \infty$. Since $\sup R\text{Hom}_A(M, A) < \infty$ by Lemma 3.2(1), we see $M^! \in D^b_{fg}(A)$, so we now prove that $M \cong M^{\dagger \dagger}$ in $D(A)$ by using induction on $g = G\text{-dim}_A M$. If $g = 0$, then $\text{Ext}_A^i(M, A) = 0$ for all $i > 0$, so $M^! \cong M^{\vee}$. Also, since $\text{Ext}^i_A^{\text{op}}(\text{Hom}_A(M, A), A) = 0$ for all $i > 0$, we have $M^{\dagger \dagger} \cong M^{\vee \vee} \cong M$. Suppose that $g \geq 1$. Let $0 \to K \to F \to M \to 0$ be an exact sequence where $F$ is a free $A$-module. Then $G\text{-dim}_A K = G\text{-dim}_A M - 1$ by Lemma 3.2(2). Thus we see that $M \cong M^{\dagger \dagger}$ in $D(A)$ by the following commutative diagram

$$
\begin{array}{ccc}
K & \longrightarrow & F \\
\downarrow^{\cong} & & \downarrow^{\cong} \\
K^{\dagger \dagger} & \longrightarrow & M^{\dagger \dagger} \\
\end{array}
$$

and using [7, IV.1, Corollary 4(a)]. Conversely, assume that $M \in \mathcal{R}(A)$. We prove $G\text{-dim}_A M < \infty$ by using induction on $s = \sup R\text{Hom}_A(M, A)$. If $s = 0$, then $\text{Ext}^i_A(M, A) = 0$ for all $i > 0$, so $M \cong M^{\dagger \dagger} \cong R\text{Hom}_A^{\text{op}}(\text{Hom}_A(M, A), A)$. Since $h^i(M) = h^i(R\text{Hom}_A^{\text{op}}(\text{Hom}_A(M, A), A)) = \text{Ext}^i_A^{\text{op}}(\text{Hom}_A(M, A), A) = 0$ for any $i > 0$, we see that $M \cong M^{\dagger \dagger} \cong M^{\vee \vee}$, so $M$ is totally reflexive. Suppose that $s \geq 1$. Let $0 \to K \to F \to M \to 0$ be an exact sequence where $F$ is a free $A$-module. Applying the functor $(-)^{\vee}$, we have $\sup R\text{Hom}_A(K, A) \leq s - 1$. Since $F, M \in \mathcal{R}(A)$, we see $K \cong K^{\dagger \dagger}$, and hence $G\text{-dim}_A K < \infty$. We get $G\text{-dim}_A M < \infty$ by Lemma 3.2(2). \qed

**Definition 3.5.** [20, Definition 2.8] Let $A$ be a noetherian connected graded algebra, and $X \in D^b_{fg}(A)$. If $X \in \mathcal{R}(A)$, then we say that $X$ has finite $G$-dimension. If $X$ has finite $G$-dimension, then we define $G$-dimension of $X$ by

$$G\text{-dim}_A X = \sup R\text{Hom}_A(X, A) < \infty.$$ 

If $X \notin \mathcal{R}(A)$, we write $G\text{-dim}_A X = \infty$.

Thus for $X \in D^b_{fg}(A)$, $G\text{-dim}_A X < \infty$ if and only if $X \in \mathcal{R}(A)$ by definition. It is easy to see that there is an inequality $G\text{-dim}_A X \leq \text{pd}_A X$, and the equality holds if $\text{pd}_A X < \infty$ (see [5, Proposition 2.3.10]).

**Lemma 3.6.** Let $X, Y \in D^b_{fg}(A)$. Then $X \oplus Y \in \mathcal{R}(A)$ if and only if $X, Y \in \mathcal{R}(A)$. In fact, $G\text{-dim}_A (X \oplus Y) = \sup \{G\text{-dim}_A X, G\text{-dim}_A Y\}$.

**Proof.** Left to the reader. \qed

Next, we define an AS-Gorenstein algebra and study relationships between the finiteness of $G$-dimension and such an algebra. “AS” stands for “Artin-Schelter” since this definition is a generalization of the notion of regular rings as introduced by Artin and Schelter [1].
Definition 3.7. A noetherian connected graded algebra $A$ is called a right AS-Gorenstein algebra if

- $\text{id}_A A = d < \infty$, and
- $\text{Ext}^i_A(k, A) \cong \begin{cases} k(\ell) & \text{for some } \ell \in \mathbb{Z} \text{ if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Left AS-Gorenstein algebras are defined similarly. We say that $A$ is AS-Gorenstein if $A$ is both right and left AS-Gorenstein.

By [17, Theorem 2.4], any noetherian connected graded algebra with finite global dimension has finite Gelfand-Kirillov dimension. Hence if $A$ is right AS-Gorenstein (in the sense defined above) of finite global dimension, then $A$ is regular in the sense of [1].

The $\chi$-condition defined below is natural and essential in noncommutative algebraic geometry (see [2]).

Definition 3.8. [2, Definition 3.2] Let $A$ be a connected graded algebra, and $M \in \text{GrMod} A$. We say that $M$ satisfies the condition $\chi_d(M)$ if $\text{Ext}^i_A(k, M)$ are finite dimensional over $k$ for all $0 \leq i \leq d$. We say that $M$ satisfies the condition $\chi(M)$ if every $\text{Ext}^i_A(k, M)$ are finite dimensional over $k$. Moreover, we say that $A$ satisfies the condition $\chi$ if $\chi(M)$ hold for all $M \in \text{grmod} A$.

Now, we give a characterization of AS-Gorenstein algebras under the assumption that algebras satisfies the $\chi$-condition. Ideas come from [12].

Proposition 3.9. Let $A$ be a noetherian connected graded algebra. Assume that $\text{depth}_A A = \text{depth}_{A^{op}} A = d$ and that $A$ satisfies the condition $\chi_d(A)$. Then the following are equivalent.

1. $A$ is AS-Gorenstein.
2. $\text{G-dim}_A M \leq d$ for any $M \in \text{grmod} A$, and $\text{G-dim}_{A^{op}} N \leq d$ for any $N \in \text{grmod} A^{op}$.

Proof. (1) $\Rightarrow$ (2): Since $A$ is AS-Gorenstein, the balanced dualizing module $\omega_A$ of $A$ (see Definition 5.8) is isomorphic to $A_{d}(\ell)$ as graded $A$-$A$ bimodules for some graded algebra automorphism $\nu \in \text{Aut}_{k} A$, so the result follows from [15, Theorem 4.7].

(2) $\Rightarrow$ (1): By lemma 3.2(1), $\text{Ext}^i_A(M, A) = 0$ for any $M \in \text{grmod} A$ and any $i > d$, so $\text{id}_A A \leq d < \infty$. Because $\text{depth}_A A = d$, we see $\text{G-dim}_A k = d$, so it follows that $\text{id}_A A = d$ and $\text{RHom}_A(k, A) \cong L[-d]$ where $L = \text{Ext}_A^d(k, A) \in \text{grmod} A^{op}$. By Lemma 3.4, we have

$$\text{RHom}_{A^{op}}(L, A) \cong \text{RHom}_{A^{op}}(\text{RHom}_A(k, A), A)[-d] \cong k[-d],$$

that is,

$$\text{Ext}^i_{A^{op}}(L, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases} \tag{3.1}$$

Since $\chi_d(A)$ holds, $\dim_k L < \infty$, so we can construct a short exact sequence

$$0 \rightarrow k(\ell) \rightarrow L \rightarrow L' \rightarrow 0$$

where $\ell = \max\{i \mid L_i \neq 0\}$. This gives a long exact sequence

$$\cdots \rightarrow \text{Ext}^d_{A^{op}}(L', A) \rightarrow \text{Ext}^d_{A^{op}}(L, A) \rightarrow \text{Ext}^d_{A^{op}}(k(-\ell), A) \rightarrow 0.$$
Since $\text{Ext}^d_{A^{\text{op}}}(k(-\ell), A) \neq 0$, (3.1) implies that $\text{Ext}^d_{A^{\text{op}}}(k, A) \cong k(-\ell)$. The assertion follows by left-right symmetry. \qed

In addition, by Zhang’s result [22], we have the following characterization.

**Proposition 3.10.** Let $A$ be a noetherian connected graded algebra satisfying the condition $\chi(A)$ on both sides. Then the following are equivalent.

1. $A$ is AS-Gorenstein.
2. There exists $n \in \mathbb{N}$ such that $G\text{-dim}_A M \leq n$ for any $M \in \text{grmod } A$, and $G\text{-dim}_A^{\text{op}} N \leq n$ for any $N \in \text{grmod } A^{\text{op}}$.

**Proof.**

(1) $\Rightarrow$ (2): The same as above.

(2) $\Rightarrow$ (1): By lemma 3.2(1), we have $\text{Ext}^i_A(M, A) = 0$ for any $M \in \text{grmod } A$ and any $i > n$, so $\text{id}_A A \leq n < \infty$. Since $A$ satisfies $\chi(A)$ on both sides, $\text{Ext}^i_A(k, A)$ and $\text{Ext}^i_{A^{\text{op}}}(k, A)$ are finite dimensional over $k$. The assertion follows by using [22, Theorem 0.3]. \qed

4. The Auslander-Bridger Formula

**Definition 4.1.** [21, Definitions 3.3, 4.1] Let $A$ be a noetherian connected graded algebra. A complex $D \in D^b(A^e)$ is called dualizing if it satisfies the following conditions:

- $\text{id}_A D < \infty$, $\text{id}_{A^{\text{op}}} D < \infty$,
- $D$ has finitely generated cohomologies over $A$ and $A^{\text{op}}$, and
- The natural morphisms $A \rightarrow \text{RHom}_A(D, D)$, and $A \rightarrow \text{RHom}_{A^{\text{op}}}(D, D)$ are isomorphisms in $D(A^e)$.

A dualizing complex $D$ over $A$ is called balanced if there are isomorphisms

$$\text{RHom}_m(D) \cong \text{RHom}_{m^{\text{op}}}(D) \cong A^*$$

in $D(A^e)$.

By [21, Proposition 3.5], if $D$ is a dualizing complex, then the functors

$$\text{RHom}_A(-, D) : D(A) \rightarrow D(A^{\text{op}}) \text{ and } \text{RHom}_{A^{\text{op}}(-, D)} : D(A^{\text{op}}) \rightarrow D(A)$$

define a duality between $D^b_{fg}(A)$ and $D^b_{fg}(A^{\text{op}})$.

Later, we will assume that $A$ has a balanced dualizing complex. By the existence theorem due to Van den Bergh [18, Theorem 6.3], we see that if $A$ admits a balanced dualizing complex, then $A$ satisfies the $\chi$-condition on both sides. In addition, we see that many graded algebras have balanced dualizing complexes. For example, by [11, Lemma 3.1 and Proposition 3.2], a graded quotient of an AS-Gorenstein algebra has a balanced dualizing complex. In particular, if $A$ is an AS-Gorenstein algebra, then $A$ has a balanced dualizing complex $A_{\nu}(-\ell)[d]$ for some graded algebra automorphism $\nu \in \text{Aut}_k A$ (see [11, Theorem 1.2]).

**Lemma 4.2.** Let $A$ be a noetherian connected graded algebra having a balanced dualizing complex $D$. For $X \in D^b_{fg}(A)$, we have $\text{depth}_A X = - \sup \text{RHom}_A(X, D)$. In particular, $\text{depth}_A A = - \sup D$.

**Proof.** This follows from [11, Proposition 4.3] and [18, Theorem 5.1]. \qed
Let $A$ be a noetherian connected graded algebra satisfying $\chi$ and $X \in D^b_{fg}(A)$ having finite projective dimension. The Auslander-Buchsbaum formula

$$pd_A X + \text{depth}_A X = \text{depth}_A A$$

has already proved by Jørgensen in [12, Theorem 3.2]. Surprisingly, Rogalski and Sierra [16] found that there is a noetherian connected graded algebra which does not satisfy the Auslander-Buchsbaum formula, that is, the $\chi$-condition is in some sense necessary for Jørgensen’s results. We now prove that the Auslander-Bridger formula also holds for the class of noncommutative noetherian connected graded algebras having balanced dualizing complexes.

**Theorem 4.3** (Auslander-Bridger formula). Let $A$ be a noetherian connected graded algebra with a balanced dualizing complex $D$. Given $X \in D^b_{fg}(A)$ with $\text{G-dim}_A X < \infty$, we have

$$\text{G-dim}_A X + \text{depth}_A X = \text{depth}_A A.$$ 

**Proof.** Since $X \in \mathcal{R}(A)$, we have

$$D \otimes_A \text{RHom}_A(X, A) \cong \text{RHom}_A(A, D) \otimes_A \text{RHom}_A(X, A)$$

by Lemma 2.2(3)

$$\cong \text{RHom}_A(X^{\oplus}, D) \cong \text{RHom}_A(A, D),$$

so it follows that

$$\text{depth}_A X = - \sup \text{RHom}_A(X, D) = - \sup (D \otimes_A \text{RHom}_A(X, A))$$

$$= - \sup D - \sup \text{RHom}_A(X, A) = \text{depth}_A A - \text{G-dim}_A X$$

by Lemma 4.2 and Lemma 2.4. \qed

**5. Characterizations of AS-Gorenstein Algebras**

Throughout this section, we assume that $A$ is a noetherian connected graded algebra with a balanced dualizing complex $D$.

**Definition 5.1.** We define the full subcategories $\mathcal{F}(A)$ and $\mathcal{I}(A)$ of $D^b_{fg}(A)$ consisting of the complexes $X \in D^b_{fg}(A)$ having finite flat and injective dimension, respectively.

Jørgensen [13, Corollary 4.6] proved that if $A$ is a noetherian connected graded algebra with a balanced dualizing complex, then $A$ is right AS-Gorenstein if and only if it is left AS-Gorenstein. Moreover, the following result was proved by Dong and Wu.

**Theorem 5.2.** [6, Theorem 3.5] The following are equivalent.

1. $A$ is AS-Gorenstein.
2. $\mathcal{F}(A) = \mathcal{I}(A)$.
3. $A \in \mathcal{I}(A)$.
4. $D \in \mathcal{F}(A)$.

In this section, we study the Auslander and Bass classes, and establish some characterizations of AS-Gorenstein algebras by using these classes and G-dimension.

**Definition 5.3.** The Auslander class $\mathcal{A}(A)$ is the full subcategory of $D^b_{fg}(A)$ consisting of the complexes $X$ satisfying
(1) $X \otimes_A^L D \in \mathcal{D}^b_{fg}(A)$, and
(2) The natural morphism $\gamma_X : X \to \mathcal{R}\text{Hom}_A(D, X \otimes_A^L D)$ is an isomorphism in $\mathcal{D}(A)$.

The Bass class $\mathcal{B}(A)$ is the full subcategory of $\mathcal{D}^b_{fg}(A)$ consisting of the complexes $X$ satisfying
(1) $\mathcal{R}\text{Hom}_A(D, X) \in \mathcal{D}^b_{fg}(A)$, and
(2) The natural morphism $\xi_X : \mathcal{R}\text{Hom}_A(D, X) \otimes_A^L D \to X$ is an isomorphism in $\mathcal{D}(A)$.

**Theorem 5.4** (Foxby equivalence). [15, Theorem 2.5] There are equivalences of categories as follows.

\[ \mathcal{A}(A) \xrightarrow{-\otimes_A^L D} \mathcal{B}(A) \]
\[ \mathcal{F}(A) \xrightarrow{-\otimes_A^L D} \mathcal{I}(A). \]

**Lemma 5.5.** $\mathcal{R}(A) = \mathcal{A}(A)$.

**Proof.** Let $X \in \mathcal{D}^b_{fg}(A)$. We have the following commutative diagram

\[ \begin{array}{ccc}
\mathcal{R}\text{Hom}_A(D, X) & \xrightarrow{\delta_X} & \mathcal{R}\text{Hom}_A(D, X \otimes_A^L D) \\
\downarrow & & \downarrow \\
\mathcal{R}\text{Hom}_A(D, X \otimes_A^L D) & \xrightarrow{\cong} & \mathcal{R}\text{Hom}_A(D, X \otimes_A^L \mathcal{R}\text{Hom}_A(A, D)) \\
\end{array} \]

so $\gamma_X$ is an isomorphism if and only if so is $\delta_X$. If $\mathcal{R}\text{Hom}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\text{op}})$, then

\[ X \otimes_A^L D \cong X \otimes_A^L \mathcal{R}\text{Hom}_A(A, D) \cong \mathcal{R}\text{Hom}_A^{\text{op}}(\mathcal{R}\text{Hom}_A(X, A), D) \]

by Lemma 2.2(3), so $X \otimes_A^L D \in \mathcal{D}^b_{fg}(A)$. Similarly, if $X \otimes_A^L D \in \mathcal{D}^b_{fg}(A)$, then

\[ \mathcal{R}\text{Hom}_A(X, A) \cong \mathcal{R}\text{Hom}_A(X, \mathcal{R}\text{Hom}_A(D, D)) \cong \mathcal{R}\text{Hom}_A(X \otimes_A^L D, D) \]

by Lemma 2.2(2), so $\mathcal{R}\text{Hom}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\text{op}})$. \qed

**Lemma 5.6.** The following hold.

(1) If $X \in \mathcal{A}(A)$, then $\mathcal{R}\text{Hom}_A(X, D) \in \mathcal{B}(A^{\text{op}})$.
(2) If $X \in \mathcal{B}(A^{\text{op}})$, then $\mathcal{R}\text{Hom}_A^{\text{op}}(X, D) \in \mathcal{A}(A)$.

**Proof.** (1): If $X \in \mathcal{A}(A)$, then $X \in \mathcal{R}(A)$ by Lemma 5.5, so we can compute

\[ \mathcal{R}\text{Hom}_A^{\text{op}}(D, \mathcal{R}\text{Hom}_A(X, D)) \cong \mathcal{R}\text{Hom}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\text{op}}) \]

by Lemma 2.2(1). The commutative diagram

\[ \begin{array}{ccc}
\mathcal{R}\text{Hom}_A(X, D) & \xrightarrow{\cong} & D \otimes_A^L \mathcal{R}\text{Hom}_A(X, A) \\
\downarrow & & \downarrow \\
\mathcal{R}\text{Hom}_A^{\text{op}}(D, \mathcal{R}\text{Hom}_A(X, D)) & \xrightarrow{\cong} & \mathcal{R}\text{Hom}_A^{\text{op}}(X, A) \otimes_A^L D \\
\end{array} \]

shows that $\xi_{\mathcal{R}\text{Hom}_A(X, D)}$ is an isomorphism. Thus $\mathcal{R}\text{Hom}_A(X, D) \in \mathcal{B}(A^{\text{op}})$.

(2): Similar to the proof of (1). \qed
We now obtain characterizations of AS-Gorenstein algebras as follows.

**Theorem 5.7.** The following are equivalent.

1. $A$ is AS-Gorenstein.
2. $\text{G-dim}_A X < \infty$ for any $X \in \mathcal{D}^b(fg)(A)$.
3. $\text{G-dim}_A M < \infty$ for any $M \in \text{grmod} A$.
4. $\text{G-dim}_A k < \infty$.
5. $\mathcal{A}(A) = \mathcal{D}^b_{fg}(A)$.
6. $\mathcal{A}^0(A) = \text{grmod} A$.
7. $k \in \mathcal{A}^0(A)$.
8. $B(A) = \mathcal{D}^b_{fg}(A)$.
9. $B^0(A) = \text{grmod} A$.
10. $k \in B^0(A)$.

(i)$^{op}$ The opposite version of (i) for $2 \leq i \leq 10$.

**Proof.** The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6) $\Rightarrow$ (7), and (8)$^{op}$ $\Rightarrow$ (9)$^{op}$ $\Rightarrow$ (10)$^{op}$ are clear. Lemma 5.5 tell us that (2), (5), and (4), (7). We prove that (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (8)$^{op}$, (4) $\Rightarrow$ (1) and (10)$^{op}$ $\Rightarrow$ (7). Since the condition (1) is left-right symmetry, the rest follows.

(1) $\Rightarrow$ (2): Since $A$ has finite right and left self-injective dimension, for any $X \in \mathcal{D}^b_{fg}(A)$, we see that $R\text{Hom}_A(X, A) \in \mathcal{D}^b_{fg}(A^{op})$ and

$$R\text{Hom}_{A^{op}}(R\text{Hom}_A(X, A), A) \cong X \otimes_A R\text{Hom}_{A^{op}}(A, A) \cong X$$

by Lemma 2.2(3).

(1) $\Rightarrow$ (8)$^{op}$: If $A$ is AS-Gorenstein, then $D \cong A_v(-\ell)[d]$ in $\mathcal{D}(A^e)$, so

$$D \otimes_A R\text{Hom}_{A^{op}}(D, X) \cong R\text{Hom}_{A^{op}}(R\text{Hom}_A(D, D), X) \cong X$$

for any $X \in \mathcal{D}^b_{fg}(A^{op})$ by Lemma 2.2(3).

(4) $\Rightarrow$ (1): If $\text{G-dim}_A k < \infty$, then

$$\text{id}_A A = \sup R\text{Hom}_A(k, A) = \text{G-dim}_A k < \infty$$

by [13, Lemma 1.10] and Lemma 3.2(1), so we have the result by Theorem 5.2.

(10)$^{op}$ $\Rightarrow$ (7): If $k \in B^0(A^{op})$, then $k \cong R\text{Hom}_{A^{op}}(k, D) \in \mathcal{A}^0(A)$ by [21, Proposition 4.4] and Lemma 5.6(2). □

Rogalski and Sierra’s example [16, Proposition 5.11] shows that there is a counterexample for (4) $\Rightarrow$ (1) if $A$ does not have a balanced dualizing complex. We remark that Theorem 5.7 does not imply Propositions 3.9 and 3.10 because the existence of a balanced dualizing complex is slightly stronger than the $\chi$-condition.

**Definition 5.8.** Let $A$ be a connected graded algebra.

1. An $A$-$A$ bimodule $\omega_A$ is called a balanced dualizing module if $\omega_A[d]$ is a balanced dualizing complex over $A$ for some $d \in \mathbb{Z}$.
2. $A$ is balanced Cohen-Macaulay if it has a balanced dualizing module.

**Lemma 5.9.** If $D \in \mathcal{A}(A)$ then $A$ is balanced Cohen-Macaulay.

**Proof.** By Lemma 5.5, $\text{G-dim}_A D < \infty$. Since $\text{depth}_A D = \inf R\text{Hom}_A(k, D) = \inf k = 0$ by [21, Proposition 4.4], it follows from the Auslander-Bridger formula...
(Theorem 4.3) that $\text{G-dim}_A D = \text{depth}_A A$. On the other hand,
\[
\inf D = \inf \text{RHom}_{A^{op}}(\text{RHom}_A(D, A), A)
\geq \inf A - \sup \text{RHom}_A(D, A) = - \text{G-dim}_A D
\]  
by Lemma 2.3. These imply that
\[- \inf D \leq \text{G-dim}_A D = \text{depth}_A A = - \sup D.
\]
Hence $\inf D = \sup D$, so $A$ is AS-Cohen-Macaulay having a balanced dualizing module $\omega_A = h^{-d}(D)$ where $d = \text{G-dim}_A D$. □

The following is the main result of this paper. This says that if the balanced dualizing complex has finite $G$-dimension, then $A$ is AS-Gorenstein. This is a generalization of [6, Theorem 3.5] (see Theorem 5.2) and a noncommutative version of [5, Theorem 3.3.5].

**Theorem 5.10.** The following are equivalent.

1. $A$ is AS-Gorenstein.
2. $\mathcal{A}(A) = \mathcal{B}(A)$.
3. $D \in \mathcal{A}(A)$.
4. $A \in \mathcal{B}(A)$.

$(i)^{op}$ The opposite version of $(i)$ for $2 \leq i \leq 4$.

**Proof.** (1) $\Rightarrow$ (2): If $A$ is AS-Gorenstein, then $\mathcal{A}(A) = \mathcal{D}_f^b(A) = \mathcal{B}(A)$ by Theorem 5.7.

(2) $\Rightarrow$ (3): Since $D \in \mathcal{I}(A) \subseteq \mathcal{B}(A)$, it follows that $D \in \mathcal{A}(A)$.

(3) $\Leftrightarrow$ (4)$^{op}$: By Lemma 5.6.

(3) $^p (4) \Rightarrow (1)$: By Lemma 5.9, $A$ is AS-Cohen-Macaulay with a balanced dualizing module $\omega_A$. Let
\[
0 \rightarrow K \rightarrow F \rightarrow \omega_A \rightarrow 0
\]  
be an exact sequence in $\text{grmod} A^{op}$, where $F$ is a free $A^{op}$-module. Since $F, \omega_A \in \mathcal{B}^0(A^{op})$, we see that $K \in \mathcal{B}^0(A^{op})$. It follows from Lemma 2.4 that
\[
\sup \text{RHom}_{A^{op}}(\omega_A, K) = \sup \omega_A + \sup \text{RHom}_{A^{op}}(\omega_A, K)
= \sup(\omega_A \otimes_A^L \text{RHom}_{A^{op}}(\omega_A, K))
= \sup(D \otimes_A^L \text{RHom}_{A^{op}}(D, K)) = \sup K = 0,
\]
so $\text{Ext}^i_{A^{op}}(\omega_A, K) = 0$ for all $i > 0$. This means that the exact sequence (5.3) splits, so $\omega_A$ is free. Hence $A$ is AS-Gorenstein by [15, Corollary 5.9].

The rest follows from left-right symmetry. □

The ideas of the next corollary come from [15, Corollary 5.6].

**Corollary 5.11.** $A$ is AS-Gorenstein if and only if there exists a totally reflexive module having finite injective dimension.

**Proof.** If $A$ is AS-Gorenstein, then $A$ is a totally reflexive module with finite injective dimension. Conversely, suppose that there exists a totally reflexive module $M$ having finite injective dimension. Since $M$ has finite injective dimension,
\[
M \cong \text{RHom}_A(D, M) \otimes_A^L D
\]
by Foxby equivalence (Theorem 5.4), so it follows that
\[ 0 = \sup M = \sup (\text{RHom}_A(D, M) \otimes^L_A D) = \sup \text{RHom}_A(D, M) - \text{depth}_A A \quad (5.4) \]
by Lemma 2.4. Moreover, \( \text{RHom}_A(D, M) \) has finite projective dimension by Foxby equivalence. We can compute
\[
\text{depth} \text{RHom}_A(D, M) = - \sup \text{RHom}_A(\text{RHom}_A(D, M), D)
= - \sup \text{RHom}_A(\text{RHom}_A^{op}(M, D), A, D)
= - \sup (\text{RHom}_A(D, A) \otimes^L_A \text{RHom}_A(M, D))
= - \sup D - \sup \text{RHom}_A(M, D)
= \text{depth}_A A + \text{depth}_A M,
\]
by Lemma 4.2, Lemma 2.2(1) and Lemma 2.4, so it follows that
\[
\text{pd}_A \text{RHom}_A(D, M) = \text{depth}_A A - \text{depth}_A \text{RHom}_A(D, M) \overset{(5.5)}{=} - \text{depth}_A M. \quad (5.6)
\]
On the other hand, similar to (5.2), we have
\[
\inf \text{RHom}_A(D, M) \geq - \text{pd}_A \text{RHom}_A(D, M).
\]
These imply that
\[
- \inf \text{RHom}_A(D, M) \overset{(5.7)}{\leq} \text{pd}_A \text{RHom}_A(D, M) \overset{(5.6)}{=} - \text{depth}_A M
= - \text{depth}_A A \overset{(5.4)}{=} - \sup \text{RHom}_A(D, M).
\]
Thus \( \inf \text{RHom}_A(D, M) = \sup \text{RHom}_A(D, M) \). Let \( N = h^d \text{RHom}_A(D, M) \in \text{grmod}_A \) where \( d = \text{depth}_A A \). Then \( N[-d] \cong \text{RHom}_A(D, M) \in \mathcal{D}(A) \). We see
\[
\text{pd}_A N = \sup \text{RHom}_A(N[-d], A) + d = \text{pd}_A \text{RHom}_A(D, M) + \text{depth}_A A = 0.
\]
So \( N \) is free. Put \( N = \bigoplus_i A(s_i) \). Since
\[
\bigoplus_i D(s_i)[-d] \cong (N \otimes^L_A D)[-d] \cong N[-d] \otimes^L_A D \cong \text{RHom}_A(D, M) \otimes^L_A D \cong M \in \mathcal{A}(A),
\]
we obtain \( D \in \mathcal{A}(A) \) by Lemma 3.6. Hence \( A \) is AS-Gorenstein by Theorem 5.10. \( \square \)

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References


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