GORENSTEIN DIMENSION AND AS-GORENSTEIN ALGEBRAS

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ABSTRACT. The purpose of this paper is to connect the notion of Gorenstein dimension with AS-Gorenstein algebras. In particular, we show that a noetherian connected graded algebra having a balanced dualizing complex is AS-Gorenstein if the balanced dualizing complex has finite Gorenstein dimension. As a preparation, we generalize the Auslander-Bridger formula to the class of noncommutative noetherian connected graded algebras having balanced dualizing complexes.

1. Introduction

In the late 1960s, the Gorenstein dimension (G-dimension for short) for finitely generated modules was introduced by Auslander [3] and developed by Auslander and Bridger [4]. G-dimension is a generalization of projective dimension. Moreover, they proved the following characterization of Gorenstein rings. A commutative noetherian local ring R is Gorenstein if and only if every finitely generated module over R has finite G-dimension. They also proved that G-dimension satisfies Auslander-Buchsbaum-type formula, namely, if a finitely generated module M over a commutative noetherian local ring R has finite G-dimension, then G-dimension of M is given by $\operatorname{depth} R - \operatorname{depth} M$. This formula is called the Auslander-Bridger formula.

Since then, relationships between G-dimension and Gorenstein rings have been studied deeply in commutative ring theory. For example, G-dimension for complexes with finitely generated cohomologies was studied by Yassemi [20] using reflexive complexes. The category of complexes of finite G-dimension is closely related to two important categories called the Auslander class and the Bass class, and some characterizations of Gorenstein rings were shown in terms of these categories (see [5, Chapter 3]). In another direction, Enochs and Jenda [8] defined a homological dimension called the Gorenstein projective dimension for non-finitely generated modules. They studied it when the ring is coherent or n-Gorenstein. For finitely generated modules over commutative noetherian rings, it coincides with the G-dimension. The Gorenstein homological dimensions have become an active area of research. See [5], [9] and [10] for more details.

Meanwhile, AS-Gorenstein algebras introduced by Artin and Schelter are an important class of algebras studied in noncommutative algebraic geometry (see [13], [14], [22] etc.). An AS-Gorenstein algebra is a noncommutative graded analogue of a commutative local Gorenstein ring.

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The purpose of this paper is to connect the notion of G-dimension with AS-Gorenstein algebras. In particular, we will give some characterizations of AS-Gorenstein algebras by using the finiteness of G-dimension. Indeed, we will prove the following statement.

Theorem 1.1. (See Theorem 5.7.) *Let A be a noetherian connected graded algebra having a balanced dualizing complex. Then the following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) $\operatorname{G-dim}_A X < \infty$ *for any* $X \in \mathcal{D}^b_{fg}(A)$.
(3) $\operatorname{G-dim}_A M < \infty$ *for any* $M \in \operatorname{grmod} A$.
(4) $\operatorname{G-dim}_A k < \infty$.

This is a noncommutative version of [4, Theorem 4.20] and [5, Theorem 2.3.14]. The balanced dualizing complex introduced by Yekutieli [21] plays an important role in the study of homological properties of noncommutative algebras. For example, noncommutative versions of Bass theorem and the no-holes theorem can be proved by using balanced dualizing complexes (see [11], [19]). In the commutative case, Theorem 1.1 is proved without the assumption that the (balanced) dualizing complex exists. Remarkably, however, there is a noetherian noncommutative connected graded algebra which does not satisfy (4) $\Rightarrow$ (1), so the assumption that an algebra has a balanced dualizing complex is a necessary condition.

Let A be a noetherian connected graded algebra having a balanced dualizing complex. In [6], Dong and Wu proved the following theorem. If the balanced dualizing complex of A has finite projective dimension, then A is AS-Gorenstein. Since G-dimension is a generalization of projective dimension, the following is a natural question to ask. If the balanced dualizing complex of A has finite G-dimension, then is A AS-Gorenstein? The main result of this paper is to show that the above question is true.

Theorem 1.2. (See Theorem 5.10.) *Let A be a noetherian connected graded algebra having a balanced dualizing complex D. Then the following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) $\operatorname{G-dim}_A D < \infty$.

This result can be viewed as a noncommutative version of [5, Theorem 3.3.5]. However, our proof is different from that of [5, Theorem 3.3.5] in the commutative case. [5] uses the fact that any module over a commutative ring R is automatically an R-R bimodule.

As a corollary, we see that if A admits a totally reflexive module having finite injective dimension, then A is AS-Gorenstein. To prove the main result, we generalize the Auslander-Bridger formula to the class of noncommutative noetherian connected graded algebras having balanced dualizing complexes (see Theorem 4.3).

2. Notations and Preliminaries

Throughout this paper, we fix a field k. Let A be a connected graded k-algebra, that is, $A = \bigoplus_{i \in \mathbb{N}} A_i$ such that $A_0 = k$. We write $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$ for the unique maximal homogeneous two-sided ideal of A, and we view $k = A/\mathfrak{m}$ as a graded A-module. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules, and by $\operatorname{grmod} A$ the full subcategory consisting of finitely generated graded right A-modules. Morphisms in $\operatorname{GrMod} A$ are right A-module homomorphisms preserving degrees.

For a graded module $M \in \operatorname{GrMod} A$ and an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \operatorname{GrMod} A$ and the shift $M(n) \in \operatorname{GrMod} A$ by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. For $M, N \in \operatorname{GrMod} A$, we write

$$\underline{\operatorname{Ext}}^i_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^i_{\operatorname{GrMod} A}(M, N(n)).$$

Let $\tau \in \operatorname{Aut}_k A$ be a graded algebra automorphism. For a graded right A-module $M \in \operatorname{GrMod} A$, we define a new graded right A-module $M_\tau \in \operatorname{GrMod} A$ by $M_\tau = M$ as graded vector spaces with the new right action $m * a = m\tau(a)$ for $m \in M$ and $a \in A$.

Let A, B be connected graded algebras. The category of graded left A-modules is denoted by $\operatorname{GrMod} A^{\mathrm{op}}$, where A^{op} is the opposite algebra. The category of graded B-A bimodules is denoted by $\operatorname{GrMod}(B^{\mathrm{op}} \otimes A)$. In particular, the category of graded A-A bimodules is denoted by $\operatorname{GrMod} A^e$, where $A^e = A^{\mathrm{op}} \otimes A$. For any $M \in \operatorname{GrMod}(B^{\mathrm{op}} \otimes A)$, we denote by $M^* := \underline{\operatorname{Hom}}_k(M, k)$ the Matlis dual of M. By definition, M^* has a graded A-B bimodule structure.

The derived category of $\operatorname{GrMod} A$ is denoted by $\mathcal{D}(A)$. We write for $\mathcal{D}_{fg}(A)$ the full subcategory of $\mathcal{D}(A)$ consisting of complexes whose cohomologies are all finitely generated graded A-modules. For any $X \in \mathcal{D}(A)$, we denote by $h^i(X)$ the i-th cohomology module of X, and define

$$\sup X = \sup\{i \mid h^i(X) \neq 0\} \text{ and } \inf X = \inf\{i \mid h^i(X) \neq 0\}.$$

We denote by $\mathcal{D}^b(A), \mathcal{D}^-(A), \mathcal{D}^+(A), \mathcal{D}^0(A)$ the full subcategories consisting of the complexes X such that $h^i(X) = 0$ for, respectively, $|i| \gg 0, i \gg 0, i \ll 0, i \neq 0$. For a full subcategory $\mathcal{S}(A)$ of $\mathcal{D}^b_{fg}(A)$, we set $\mathcal{S}^0(A) = \mathcal{S}(A) \cap \mathcal{D}^0(A)$. We identify $\mathcal{D}^0(A)$ with $\operatorname{GrMod} A$, and $\mathcal{D}^0_{fg}(A)$ with $\operatorname{grmod} A$. For $X \in \mathcal{D}(A)$ and an integer $n \in \mathbb{Z}$, the twist of X is denoted by $X[n] \in \mathcal{D}(A)$ so that $(X[n])^i = X^{n+i}$.

The derived functors of $\underline{\operatorname{Hom}}_A(-,-)$ and $- \otimes_A -$ are denoted $\mathrm{R}\underline{\operatorname{Hom}}_A(-,-)$ and $- \otimes^{\mathrm{L}}_A -$. As usual, we define

$$\underline{\operatorname{Ext}}^i_A(-,-) = h^i(\mathrm{R}\underline{\operatorname{Hom}}_A(-,-)) \text{ and } \operatorname{Tor}^A_{-i}(-,-) = h^i(- \otimes^{\mathrm{L}}_A -).$$

See [21], [12] or [6] for details. Let $\mathfrak{m} = A_{\geq 1}$. We define the functor

$$\underline{\Gamma}_{\mathfrak{m}}(-) : \operatorname{GrMod}(B^{\mathrm{op}} \otimes A) \to \operatorname{GrMod}(B^{\mathrm{op}} \otimes A), \quad M \mapsto \lim_{n \to \infty} \underline{\operatorname{Hom}}_A(A/A_{\geq n}, M).$$

Its right derived functor is denoted by $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-)$. See [21] for details.

Definition 2.1. Let A be a connected graded algebra. For each $X \in \mathcal{D}^+(A)$, we define

$$\operatorname{depth}_A X = \inf \mathrm{R}\underline{\operatorname{Hom}}_A(k, X) = \inf\{i \mid \underline{\operatorname{Ext}}^i_A(k, X) \neq 0\}$$

We will often use the following hyperhomological isomorphisms. Let A, B, R, S be noetherian connected graded algebras.

Lemma 2.2. (cf. [13, Theorems 1.2 to 1.4])

(1) *Let $X \in \mathcal{D}^-(S^{\mathrm{op}} \otimes A), Y \in \mathcal{D}^-(B^{\mathrm{op}} \otimes R), Z \in \mathcal{D}^+(B^{\mathrm{op}} \otimes A)$. Then there is a natural isomorphism in $\mathcal{D}^+(R^{\mathrm{op}} \otimes S)$,*

$$\mathrm{R}\underline{\operatorname{Hom}}_A(X, \mathrm{R}\underline{\operatorname{Hom}}_{B^{\mathrm{op}}}(Y, Z)) \cong \mathrm{R}\underline{\operatorname{Hom}}_{B^{\mathrm{op}}}(Y, \mathrm{R}\underline{\operatorname{Hom}}_A(X, Z)).$$

(2) *Let* $X \in \mathcal{D}^-(S^{\rm op} \otimes A), Y \in \mathcal{D}^-(A^{\rm op} \otimes B), Z \in \mathcal{D}^+(R^{\rm op} \otimes B)$. *Then there is a natural isomorphism in* $\mathcal{D}^+(R^{\rm op} \otimes S)$,

$$\mathrm{R}\underline{\mathrm{Hom}}_B(X \otimes^{\mathrm{L}}_A Y, Z) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(X, \mathrm{R}\underline{\mathrm{Hom}}_B(Y, Z)).$$

(3) *Let* $X \in \mathcal{D}^-_{fg}(A^{\rm op}), Y \in \mathcal{D}^b(A^{\rm op} \otimes B), Z \in \mathcal{D}^+(R^{\rm op} \otimes B)$. *If either* $\mathrm{pd}_{A^{\rm op}} X < \infty$ *or* $\mathrm{id}_B Z < \infty$, *then there is a natural isomorphism in* $\mathcal{D}(R^{\rm op})$,

$$\mathrm{R}\underline{\mathrm{Hom}}_B(Y, Z) \otimes^{\mathrm{L}}_A X \cong \mathrm{R}\underline{\mathrm{Hom}}_B(\mathrm{R}\underline{\mathrm{Hom}}_{A^{\rm op}}(X, Y), Z).$$

Lemma 2.3. (cf. [15, Lemma 1.3]) *If* $X \in \mathcal{D}^-(B^{\rm op} \otimes A), Y \in \mathcal{D}^+(C^{\rm op} \otimes A)$, *then*

$$\inf \mathrm{R}\underline{\mathrm{Hom}}_A(X, Y) \geq \inf Y - \sup X.$$

Lemma 2.4. (cf. [13, Lemma 1.8]) *If* $X \in \mathcal{D}^-(B^{\rm op} \otimes A), Y \in \mathcal{D}^-(A^{\rm op} \otimes C)$, *then*

$$\sup X \otimes^{\mathrm{L}}_A Y \leq \sup X + \sup Y.$$

Moreover, if $h^{\sup X} X, h^{\sup Y} Y$ *are left bounded, then* $\sup X \otimes^{\mathrm{L}}_A Y = \sup X + \sup Y$.

3. Gorenstein Dimension

First, we recall the definition of G-dimension of finitely generated modules and of bounded complexes with finitely generated cohomologies. Throughout this section, A is a noetherian connected graded algebra.

If $M \in \mathrm{GrMod}\, A$, then we define $M^\vee = \underline{\mathrm{Hom}}_A(M, A) \in \mathrm{GrMod}\, A^{\rm op}$. Similarly, if $N \in \mathrm{GrMod}\, A^{\rm op}$, then we define $N^\vee = \underline{\mathrm{Hom}}_{A^{\rm op}}(N, A) \in \mathrm{GrMod}\, A$.

Definition 3.1. (cf. [4, Chapter 3], [5, Definition 1.2.3]) Let A be a noetherian connected graded algebra.

(1) We say that $M \in \mathrm{grmod}\, A$ is totally reflexive if
   (a) The natural homomorphism $M \to M^{\vee\vee}$ is an isomorphism.
   (b) $\underline{\mathrm{Ext}}^i_A(M, A) = \underline{\mathrm{Ext}}^i_{A^{\rm op}}(M^\vee, A) = 0$ for all $i > 0$.

(2) Let $M \in \mathrm{grmod}\, A$. If there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of graded right A-modules such that each G_i is totally reflexive, then we say that M has G-dimension at most n. If such an integer n does not exist, then we say that M has infinite G-dimension, and write $\text{G-dim}_A M = \infty$. If M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has G-dimension n, and write $\text{G-dim}_A M = n$. We set $\text{G-dim}_A 0 = -\infty$.

Lemma 3.2. *The following hold.*

(1) *Let* $M \in \mathrm{grmod}\, A$. *If* $\text{G-dim}_A M < \infty$, *then*

$$\text{G-dim}_A M = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(M, A) = \sup\{i \mid \underline{\mathrm{Ext}}^i_A(M, A) \neq 0\}.$$

(2) *Let* $0 \to K \to G \to M \to 0$ *be an exact sequence in* $\mathrm{grmod}\, A$. *If* $\text{G-dim}_A G = 0$, *then* $\text{G-dim}_A K = \sup\{\text{G-dim}_A M - 1, 0\}$.

Proof. See [5, Theorem 1.2.7] and [5, Corollary 1.2.9]. □

If $X \in \mathcal{D}^b(A)$, then we define $X^\dagger = \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \in \mathcal{D}(A^{\rm op})$. Similarly, if $Y \in \mathcal{D}^b(A^{\rm op})$, then we define $Y^\dagger = \mathrm{R}\underline{\mathrm{Hom}}_{A^{\rm op}}(Y, A) \in \mathcal{D}(A)$.

Definition 3.3. The reflexive class $\mathcal{R}(A)$ is the full subcategory of $\mathcal{D}^b_{fg}(A)$ consisting of the complexes X satisfying

(1) $X^\dagger = \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\mathrm{op}})$, and
(2) The natural morphism $X \to X^{\dagger\dagger}$ is an isomorphism in $\mathcal{D}(A)$.

Lemma 3.4. *Let* $M \in \operatorname{grmod} A$. *Then* $\operatorname{G-dim}_A M < \infty$ *if and only if* $M \in \mathcal{R}(A)$.

Proof. We prove this along the same lines as in [20, Theorem 2.7]. Assume that $\operatorname{G-dim}_A M < \infty$. Since $\sup \mathrm{R}\underline{\mathrm{Hom}}_A(M, A) < \infty$ by Lemma 3.2(1), we see $M^\dagger \in \mathcal{D}^b_{fg}(A)$, so we now prove that $M \cong M^{\dagger\dagger}$ in $\mathcal{D}(A)$ by using induction on $g = \operatorname{G-dim}_A M$. If $g = 0$, then $\underline{\mathrm{Ext}}^i_A(M, A) = 0$ for all $i > 0$, so $M^\dagger \cong M^\vee$. Also since $\underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}(\underline{\mathrm{Hom}}_A(M, A), A) = 0$ for all $i > 0$, we have $M^{\dagger\dagger} \cong M^{\vee\vee} \cong M$. Suppose that $g \geq 1$. Let $0 \to K \to F \to M \to 0$ be an exact sequence where F is a free A-module. Then $\operatorname{G-dim}_A K = \operatorname{G-dim}_A M - 1$ by Lemma 3.2(2). Thus we see that $M \cong M^{\dagger\dagger}$ in $\mathcal{D}(A)$ by the following commutative diagram

$$\begin{array}{ccccccc} K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & K[1] \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong \\ K^{\dagger\dagger} & \to & F^{\dagger\dagger} & \to & M^{\dagger\dagger} & \to & K^{\dagger\dagger}[1] \end{array}$$

and using [7, IV.1, Corollary 4(a)]. Conversely, assume that $M \in \mathcal{R}(A)$. We prove $\operatorname{G-dim}_A M < \infty$ by using induction on $s = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(M, A)$. If $s = 0$, then $\underline{\mathrm{Ext}}^i_A(M, A) = 0$ for all $i > 0$, so $M \cong M^{\dagger\dagger} \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\underline{\mathrm{Hom}}_A(M, A), A)$. Since

$$h^i(M) = h^i(\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\underline{\mathrm{Hom}}_A(M, A), A)) = \underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}(\underline{\mathrm{Hom}}_A(M, A), A)) = 0$$

for any $i > 0$, we see that $M \cong M^{\dagger\dagger} \cong M^{\vee\vee}$, so M is totally reflexive. Suppose that $s \geq 1$. Let $0 \to K \to F \to M \to 0$ be an exact sequence where F is a free A-module. Applying the functor $(-)^\vee$, we have $\sup \mathrm{R}\underline{\mathrm{Hom}}_A(K, A) \leq s - 1$. Since $F, M \in \mathcal{R}(A)$, we see $K \cong K^{\dagger\dagger}$, and hence $\operatorname{G-dim}_A K < \infty$. We get $\operatorname{G-dim}_A M < \infty$ by Lemma 3.2(2). □

Definition 3.5. [20, Definition 2.8] Let A be a noetherian connected graded algebra, and $X \in \mathcal{D}^b_{fg}(A)$. If $X \in \mathcal{R}(A)$, then we say that X has finite G-dimension. If X has finite G-dimension, then we define G-dimension of X by

$$\operatorname{G-dim}_A X = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) < \infty.$$

If $X \notin \mathcal{R}(A)$, we write $\operatorname{G-dim}_A X = \infty$.

Thus for $X \in \mathcal{D}^b_{fg}(A)$, $\operatorname{G-dim}_A X < \infty$ if and only if $X \in \mathcal{R}(A)$ by definition. It is easy to see that there is an inequality $\operatorname{G-dim}_A X \leq \operatorname{pd}_A X$, and the equality holds if $\operatorname{pd}_A X < \infty$ (see [5, Proposition 2.3.10]).

Lemma 3.6. *Let* $X, Y \in \mathcal{D}^b_{fg}(A)$. *Then* $X \oplus Y \in \mathcal{R}(A)$ *if and only if* $X, Y \in \mathcal{R}(A)$. *In fact,* $\operatorname{G-dim}_A(X \oplus Y) = \sup\{\operatorname{G-dim}_A X, \operatorname{G-dim}_A Y\}$.

Proof. Left to the reader. □

Next, we define an AS-Gorenstein algebra and study relationships between the finiteness of G-dimension and such an algebra. "AS" stands for "Artin-Schelter" since this definition is a generalization of the notion of regular rings as introducd by Artin and Schelter [1].

Definition 3.7. A noetherian connected graded algebra A is called a right AS-Gorenstein algebra if

- $\operatorname{id}_A A = d < \infty$, and
- $\underline{\operatorname{Ext}}^i_A(k, A) \cong \begin{cases} k(\ell) & \text{for some } \ell \in \mathbb{Z} \quad \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Left AS-Gorenstein algebras are defined similarly. We say that A is AS-Gorenstein if A is both right and left AS-Gorenstein.

By [17, Theorem 2.4], any noetherian connected graded algebra with finite global dimension has finite Gelfand-Kirillov dimension. Hence if A is right AS-Gorenstein (in the sense defined above) of finite global dimension, then A is regular in the sense of [1].

The χ-condition defined below is natural and essential in noncommutative algebraic geometry (see [2]).

Definition 3.8. [2, Definition 3.2] Let A be a connected graded algebra, and $M \in \operatorname{GrMod} A$. We say that M satisfies the condition $\chi_d(M)$ if $\underline{\operatorname{Ext}}^i_A(k, M)$ are finite dimensional over k for all $0 \leq i \leq d$. We say that M satisfies the condition $\chi(M)$ if every $\underline{\operatorname{Ext}}^i_A(k, M)$ are finite dimensional over k. Moreover, we say that A satisfies the condition χ if $\chi(M)$ hold for all $M \in \operatorname{grmod} A$.

Now, we give a characterization of AS-Gorenstein algebras under the assumption that algebras satisfies the χ-condition. Ideas come from [12].

Proposition 3.9. *Let A be a noetherian connected graded algebra. Assume that* $\operatorname{depth}_A A = \operatorname{depth}_{A^{\mathrm{op}}} A = d$ *and that A satisfies the condition $\chi_d(A)$. Then the following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) $\operatorname{G-dim}_A M \leq d$ *for any* $M \in \operatorname{grmod} A$, *and* $\operatorname{G-dim}_{A^{\mathrm{op}}} N \leq d$ *for any* $N \in \operatorname{grmod} A^{\mathrm{op}}$.

Proof. (1) $\Rightarrow$ (2): Since A is AS-Gorenstein, the balanced dualizing module ω_A of A (see Definition 5.8) is isomorphic to $A_\nu(-\ell)$ as graded A-A bimodules for some graded algebra automorphism $\nu \in \operatorname{Aut}_k A$, so the result follows from [15, Theorem 4.7].

(2) $\Rightarrow$ (1): By lemma 3.2(1), $\underline{\operatorname{Ext}}^i_A(M, A) = 0$ for any $M \in \operatorname{grmod} A$ and any $i > d$, so $\operatorname{id}_A A \leq d < \infty$. Because $\operatorname{depth}_A A = d$, we see $\operatorname{G-dim}_A k = d$, so it follows that $\operatorname{id}_A A = d$ and $\mathrm{R}\underline{\operatorname{Hom}}_A(k, A) \cong L[-d]$ where $L = \underline{\operatorname{Ext}}^d_A(k, A) \in \operatorname{grmod} A^{\mathrm{op}}$. By Lemma 3.4, we have

$$\mathrm{R}\underline{\operatorname{Hom}}_{A^{\mathrm{op}}}(L, A) \cong \mathrm{R}\underline{\operatorname{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\operatorname{Hom}}_A(k, A), A)[-d] \cong k[-d],$$

that is,

$$\underline{\operatorname{Ext}}^i_{A^{\mathrm{op}}}(L, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases} \tag{3.1}$$

Since $\chi_d(A)$ holds, $\dim_k L < \infty$, so we can construct a short exact sequence

$$0 \to k(-\ell) \to L \to L' \to 0$$

where $\ell = \max\{i \mid L_i \neq 0\}$. This gives a long exact sequence

$$\cdots \to \underline{\operatorname{Ext}}^d_{A^{\mathrm{op}}}(L', A) \to \underline{\operatorname{Ext}}^d_{A^{\mathrm{op}}}(L, A) \to \underline{\operatorname{Ext}}^d_{A^{\mathrm{op}}}(k(-\ell), A) \to 0\ .$$

Since $\underline{\mathrm{Ext}}^d_{A^{\mathrm{op}}}(k(-\ell), A) \neq 0$, (3.1) implies that $\underline{\mathrm{Ext}}^d_{A^{\mathrm{op}}}(k, A) \cong k(-\ell)$. The assertion follows by left-right symmetry. □

In addition, by Zhang's result [22], we have the following characterization.

Proposition 3.10. *Let A be a noetherian connected graded algebra satisfying the condition $\chi(A)$ on both sides. Then the following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) *There exists $n \in \mathbb{N}$ such that $\operatorname{G-dim}_A M \leq n$ for any $M \in \operatorname{grmod} A$, and $\operatorname{G-dim}_{A^{\mathrm{op}}} N \leq n$ for any $N \in \operatorname{grmod} A^{\mathrm{op}}$.*

Proof. (1) ⇒ (2): The same as above.

(2) ⇒ (1): By lemma 3.2(1), we have $\underline{\mathrm{Ext}}^i_A(M, A) = 0$ for any $M \in \operatorname{grmod} A$ and any $i > n$, so $\operatorname{id}_A A \leq n < \infty$. Since A satisfies $\chi(A)$ on both sides, $\underline{\mathrm{Ext}}^i_A(k, A)$ and $\underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}(k, A)$ are finite dimensional over k. The assertion follows by using [22, Theorem 0.3]. □

4. The Auslander-Bridger Formula

Definition 4.1. [21, Definitions 3.3, 4.1] Let A be a noetherian connected graded algebra. A complex $D \in \mathcal{D}^b(A^e)$ is called dualizing if it satisfies the following conditions:

- $\operatorname{id}_A D < \infty, \operatorname{id}_{A^{\mathrm{op}}} D < \infty$,
- D has finitely generated cohomologies over A and A^{op}, and
- The natural morphisms $A \to \mathrm{R}\underline{\mathrm{Hom}}_A(D, D)$, and $A \to \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(D, D)$ are isomorphisms in $\mathcal{D}(A^e)$.

A dualizing complex D over A is called balanced if there are isomorphisms

$$\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(D) \cong \mathrm{R}\underline{\Gamma}_{\mathfrak{m}^{\mathrm{op}}}(D) \cong A^*$$

in $\mathcal{D}(A^e)$.

By [21, Proposition 3.5], if D is a dualizing complex, then the functors

$$\mathrm{R}\underline{\mathrm{Hom}}_A(-, D) : \mathcal{D}(A) \to \mathcal{D}(A^{\mathrm{op}}) \text{ and } \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(-, D) : \mathcal{D}(A^{\mathrm{op}}) \to \mathcal{D}(A)$$

define a duality between $\mathcal{D}^b_{fg}(A)$ and $\mathcal{D}^b_{fg}(A^{\mathrm{op}})$.

Later, we will assume that A has a balanced dualizing complex. By the existence theorem due to Van den Bergh [18, Theorem 6.3], we see that if A admits a balanced dualizing complex, then A satisfies the χ-condition on both sides. In addition, we see that many graded algebras have balanced dualizing complexes. For example, by [11, Lemma 3.1 and Proposition 3.2], a graded quotient of an AS-Gorenstein algebra has a balanced dualizing complex. In particular, if A is an AS-Gorenstein algebra, then A has a balanced dualizing complex $A_\nu(-\ell)[d]$ for some graded algebra automorphism $\nu \in \operatorname{Aut}_k A$ (see [11, Theorem 1.2]).

Lemma 4.2. *Let A be a noetherian connected graded algebra having a balanced dualizing complex D. For $X \in \mathcal{D}^b_{fg}(A)$, we have $\operatorname{depth}_A X = -\sup \mathrm{R}\underline{\mathrm{Hom}}_A(X, D)$. In particular, $\operatorname{depth}_A A = -\sup D$.*

Proof. This follows from [11, Proposition 4.3] and [18, Theorem 5.1]. □

Let A be a noetherian connected graded algebra satisfying χ and $X \in \mathcal{D}^b_{fg}(A)$ having finite projective dimension. The Auslander-Buchsbaum formula

$$\operatorname{pd}_A X + \operatorname{depth}_A X = \operatorname{depth}_A A$$

has already proved by Jørgensen in [12, Theorem 3.2]. Surprisingly, Rogalski and Sierra [16] found that there is a noetherian connected graded algebra which does not satisfy the Auslander-Buchsbaum formula, that is, the χ-condition is in some sense necessary for Jørgensen's results. We now prove that the Auslander-Bridger formula also holds for the class of noncommutative noetherian connected graded algebras having balanced dualizing complexes.

Theorem 4.3 (Auslander-Bridger formula)**.** *Let A be a noetherian connected graded algebra with a balanced dualizing complex D. Given $X \in \mathcal{D}^b_{fg}(A)$ with $\operatorname{G-dim}_A X < \infty$, we have*

$$\operatorname{G-dim}_A X + \operatorname{depth}_A X = \operatorname{depth}_A A.$$

Proof. Since $X \in \mathcal{R}(A)$, we have

$$\begin{aligned} D \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) &\xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_A(A, D) \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \\ &\xrightarrow[\text{Lem2.2(3)}]{\cong} \mathrm{R}\underline{\mathrm{Hom}}_A(X^{\dagger\dagger}, D) \xrightarrow[\mathrm{R}\underline{\mathrm{Hom}}_A(\delta_X, D)]{\cong} \mathrm{R}\underline{\mathrm{Hom}}_A(X, D), \end{aligned} \tag{4.1}$$

so it follows that

$$\begin{aligned} \operatorname{depth}_A X &= -\sup \mathrm{R}\underline{\mathrm{Hom}}_A(X, D) = -\sup(D \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(X, A)) \\ &= -\sup D - \sup \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) = \operatorname{depth}_A A - \operatorname{G-dim}_A X \end{aligned}$$

by Lemma 4.2 and Lemma 2.4. □

5. Characterizations of AS-Gorenstein Algebras

Throughout this section, we assume that A is a noetherian connected graded algebra with a balanced dualizing complex D.

Definition 5.1. We define the full subcategories $\mathcal{F}(A)$ and $\mathcal{I}(A)$ of $\mathcal{D}^b_{fg}(A)$ consisting of the complexes $X \in \mathcal{D}^b_{fg}(A)$ having finite flat and injective dimension, respectively.

Jørgensen [13, Corollary 4.6] proved that if A is a noetherian connected graded algebra with a balanced dualizing complex, then A is right AS-Gorenstein if and only if it is left AS-Gorenstein. Moreover, the following result was proved by Dong and Wu.

Theorem 5.2. [6, Theorem 3.5] *The following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) *$\mathcal{F}(A) = \mathcal{I}(A)$.*
(3) *$A \in \mathcal{I}(A)$.*
(4) *$D \in \mathcal{F}(A)$.*

In this section, we study the Auslander and Bass classes, and establish some characterizations of AS-Gorenstein algebras by using these classes and G-dimension.

Definition 5.3. The Auslander class $\mathcal{A}(A)$ is the full subcategory of $\mathcal{D}^b_{fg}(A)$ consisting of the complexes X satisfying

(1) $X \otimes^{\mathrm{L}}_A D \in \mathcal{D}^b_{fg}(A)$, and
(2) The natural morphism $\gamma_X : X \to \mathrm{R}\underline{\mathrm{Hom}}_A(D, X \otimes^{\mathrm{L}}_A D)$ is an isomorphism in $\mathcal{D}(A)$.

The Bass class $\mathcal{B}(A)$ is the full subcategory of $\mathcal{D}^b_{fg}(A)$ consisting of the complexes X satisfying

(1) $\mathrm{R}\underline{\mathrm{Hom}}_A(D, X) \in \mathcal{D}^b_{fg}(A)$, and
(2) The natural morphism $\xi_X : \mathrm{R}\underline{\mathrm{Hom}}_A(D, X) \otimes^{\mathrm{L}}_A D \to X$ is an isomorphism in $\mathcal{D}(A)$.

Theorem 5.4 (Foxby equivalence)**.** [15, Theorem 2.5] *There are equivalences of categories as follows.*

$$\begin{array}{ccc} \mathcal{A}(A) & \underset{\mathrm{R}\underline{\mathrm{Hom}}_A(D,-)}{\overset{-\otimes^{\mathrm{L}}_A D}{\rightleftarrows}}\ (\sim) & \mathcal{B}(A) \\ \cup\!\mid & & \cup\!\mid \\ \mathcal{F}(A) & \underset{\mathrm{R}\underline{\mathrm{Hom}}_A(D,-)}{\overset{-\otimes^{\mathrm{L}}_A D}{\rightleftarrows}}\ (\sim) & \mathcal{I}(A). \end{array}$$

Lemma 5.5. $\mathcal{R}(A) = \mathcal{A}(A)$.

Proof. Let $X \in \mathcal{D}^b_{fg}(A)$. We have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\gamma_X} & \mathrm{R}\underline{\mathrm{Hom}}_A(D, X \otimes^{\mathrm{L}}_A D) & \xrightarrow{\cong} & \mathrm{R}\underline{\mathrm{Hom}}_A(D, X \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(A, D)) \\ {\scriptstyle \delta_X}\downarrow & & & & \downarrow{\scriptstyle \cong\ \mathrm{Lem2.2(3)}} \\ X^{\dagger\dagger} & & \xleftarrow[\mathrm{Lem2.2(1)}]{\cong} & & \mathrm{R}\underline{\mathrm{Hom}}_A(D, \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\mathrm{Hom}}_A(X, A), D)), \end{array}$$

so γ_X is an isomorphism if and only if so is δ_X. If $\mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\mathrm{op}})$, then

$$X \otimes^{\mathrm{L}}_A D \cong X \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(A, D) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\mathrm{Hom}}_A(X, A), D)$$

by Lemma 2.2(3), so $X \otimes^{\mathrm{L}}_A D \in \mathcal{D}^b_{fg}(A)$. Similarly, if $X \otimes^{\mathrm{L}}_A D \in \mathcal{D}^b_{fg}(A)$, then

$$\mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(X, \mathrm{R}\underline{\mathrm{Hom}}_A(D, D)) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(X \otimes^{\mathrm{L}}_A D, D)$$

by Lemma 2.2(2), so $\mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\mathrm{op}})$. ☐

Lemma 5.6. *The following hold.*

(1) *If* $X \in \mathcal{A}(A)$*, then* $\mathrm{R}\underline{\mathrm{Hom}}_A(X, D) \in \mathcal{B}(A^{\mathrm{op}})$.
(2) *If* $X \in \mathcal{B}(A^{\mathrm{op}})$*, then* $\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(X, D) \in \mathcal{A}(A)$.

Proof. (1): If $X \in \mathcal{A}(A)$, then $X \in \mathcal{R}(A)$ by Lemma 5.5, so we can compute

$$\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(D, \mathrm{R}\underline{\mathrm{Hom}}_A(X, D)) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\mathrm{op}}) \tag{5.1}$$

by Lemma 2.2(1). The commutative diagram

$$\begin{array}{ccc} \mathrm{R}\underline{\mathrm{Hom}}_A(X, D) & \xleftarrow[(4.1)]{\cong} & D \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \\ & {\scriptstyle \xi_{\mathrm{R}\underline{\mathrm{Hom}}_A(X,D)}}\nwarrow \quad \swarrow{\scriptstyle \cong\ (5.1)} & \\ & D \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(D, \mathrm{R}\underline{\mathrm{Hom}}_A(X, D)) & \end{array}.$$

shows that $\xi_{\mathrm{R}\underline{\mathrm{Hom}}_A(X,D)}$ is an isomorphism. Thus $\mathrm{R}\underline{\mathrm{Hom}}_A(X, D) \in \mathcal{B}(A^{\mathrm{op}})$.

(2): Similar to the proof of (1). ☐

We now obtain characterizations of AS-Gorenstein algebras as follows.

Theorem 5.7. *The following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) $\operatorname{G-dim}_A X < \infty$ *for any* $X \in \mathcal{D}^b_{fg}(A)$.
(3) $\operatorname{G-dim}_A M < \infty$ *for any* $M \in \operatorname{grmod} A$.
(4) $\operatorname{G-dim}_A k < \infty$.
(5) $\mathcal{A}(A) = \mathcal{D}^b_{fg}(A)$.
(6) $\mathcal{A}^0(A) = \operatorname{grmod} A$.
(7) $k \in \mathcal{A}^0(A)$.
(8) $\mathcal{B}(A) = \mathcal{D}^b_{fg}(A)$.
(9) $\mathcal{B}^0(A) = \operatorname{grmod} A$.
(10) $k \in \mathcal{B}^0(A)$.
$(i)^{\mathrm{op}}$ *The opposite version of* (i) *for* $2 \le i \le 10$.

Proof. The implications (2) ⇒ (3) ⇒ (4), (5) ⇒ (6) ⇒ (7), and $(8)^{\mathrm{op}} \Rightarrow (9)^{\mathrm{op}} \Rightarrow (10)^{\mathrm{op}}$ are clear. Lemma 5.5 tell us that (2) ⇔ (5) and (4) ⇔ (7). We prove that (1) ⇒ (2), (1) ⇒ $(8)^{\mathrm{op}}$, (4) ⇒ (1) and $(10)^{\mathrm{op}} \Rightarrow (7)$. Since the condition (1) is left-right symmetry, the rest follows.

(1) ⇒ (2): Since A has finite right and left self-injective dimension, for any $X \in \mathcal{D}^b_{fg}(A)$, we see that $\mathrm{R}\underline{\mathrm{Hom}}_A(X, A) \in \mathcal{D}^b_{fg}(A^{\mathrm{op}})$ and

$$\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\mathrm{Hom}}_A(X, A), A) \cong X \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(A, A) \cong X$$

by Lemma 2.2(3).

(1) ⇒ $(8)^{\mathrm{op}}$: If A is AS-Gorenstein, then $D \cong A_\nu(-\ell)[d]$ in $\mathcal{D}(A^e)$, so

$$D \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(D, X) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\mathrm{Hom}}_A(D, D), X) \cong X$$

for any $X \in \mathcal{D}^b_{fg}(A^{\mathrm{op}})$ by Lemma 2.2(3).

(4) ⇒ (1): If $\operatorname{G-dim}_A k < \infty$, then

$$\mathrm{id}_A A = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(k, A) = \operatorname{G-dim}_A k < \infty$$

by [13, Lemma 1.10] and Lemma 3.2(1), so we have the result by Theorem 5.2.

$(10)^{\mathrm{op}} \Rightarrow (7)$: If $k \in \mathcal{B}^0(A^{\mathrm{op}})$, then $k \cong \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(k, D) \in \mathcal{A}^0(A)$ by [21, Proposition 4.4] and Lemma 5.6(2). □

Rogalski and Sierra's example [16, Proposition 5.11] shows that there is a counterexample for (4) ⇒ (1) if A does not have a balanced dualizing complex. We remark that Theorem 5.7 does not imply Propositions 3.9 and 3.10 because the existence of a balanced dualizing complex is slightly stronger than the χ-condition.

Definition 5.8. Let A be a connected graded algebra.

(1) An A-A bimodule ω_A is called a balanced dualizing module if $\omega_A[d]$ is a balanced dualizing complex over A for some $d \in \mathbb{Z}$.
(2) A is balanced Cohen-Macaulay if it has a balanced dualizing module.

Lemma 5.9. *If $D \in \mathcal{A}(A)$ then A is balanced Cohen-Macaulay.*

Proof. By Lemma 5.5, $\operatorname{G-dim}_A D < \infty$. Since $\operatorname{depth}_A D = \inf \mathrm{R}\underline{\mathrm{Hom}}_A(k, D) = \inf k = 0$ by [21, Proposition 4.4], It follows from the Auslander-Bridger formula

(Theorem 4.3) that $\text{G-dim}_A D = \operatorname{depth}_A A$. On the other hand,

$$\begin{aligned}\inf D &= \inf \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\mathrm{Hom}}_A(D, A), A) \\ &\geq \inf A - \sup \mathrm{R}\underline{\mathrm{Hom}}_A(D, A) = -\,\text{G-dim}_A D \end{aligned} \tag{5.2}$$

by Lemma 2.3. These imply that

$$-\inf D \leq \text{G-dim}_A D = \operatorname{depth}_A A = -\sup D.$$

Hence $\inf D = \sup D$, so A is AS-Cohen-Macaulay having a balanced dualizing module $\omega_A = h^{-d}(D)$ where $d = \text{G-dim}_A D$. □

The following is the main result of this paper. This says that if the balanced dualizing complex has finite G-dimension, then A is AS-Gorenstein. This is a generalization of [6, Theorem 3.5] (see Theorem 5.2) and a noncommutative version of [5, Theorem 3.3.5].

Theorem 5.10. *The following are equivalent.*

(1) *A is AS-Gorenstein.*
(2) *$\mathcal{A}(A) = \mathcal{B}(A)$.*
(3) *$D \in \mathcal{A}(A)$.*
(4) *$A \in \mathcal{B}(A)$.*
$(i)^{\mathrm{op}}$ *The opposite version of (i) for $2 \leq i \leq 4$.*

Proof. (1) ⇒ (2): If A is AS-Gorenstein, then $\mathcal{A}(A) = \mathcal{D}^b_{fg}(A) = \mathcal{B}(A)$ by Theorem 5.7.

(2) ⇒ (3): Since $D \in \mathcal{I}(A) \subseteq \mathcal{B}(A)$, it follows that $D \in \mathcal{A}(A)$.

(3) ⇔ $(4)^{\mathrm{op}}$: By Lemma 5.6.

(3) + $(4)^{\mathrm{op}}$ ⇒ (1): By Lemma 5.9, A is AS-Cohen-Macaulay with a balanced dualizing module ω_A. Let

$$0 \to K \to F \to \omega_A \to 0 \tag{5.3}$$

be an exact sequence in $\operatorname{grmod} A^{\mathrm{op}}$, where F is a free A^{op}-module. Since $F, \omega_A \in \mathcal{B}^0(A^{\mathrm{op}})$, we see that $K \in \mathcal{B}^0(A^{\mathrm{op}})$. It follows from Lemma 2.4 that

$$\begin{aligned}\sup \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\omega_A, K) &= \sup \omega_A + \sup \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\omega_A, K) \\ &= \sup(\omega_A \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\omega_A, K)) \\ &= \sup(D \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(D, K)) = \sup K = 0,\end{aligned}$$

so $\underline{\mathrm{Ext}}^i_{A^{\mathrm{op}}}(\omega_A, K) = 0$ for all $i > 0$. This means that the exact sequence (5.3) splits, so ω_A is free. Hence A is AS-Gorenstein by [15, Corollary 5.9].

The rest follows from left-right symmetry. □

The ideas of the next corollary come from [15, Corollary 5.6].

Corollary 5.11. *A is AS-Gorenstein if and only if there exists a totally reflexive module having finite injective dimension.*

Proof. If A is AS-Gorenstein, then A is a totally reflexive module with finite injective dimension. Conversely, suppose that there exists a totally reflexive module M having finite injective dimension. Since M has finite injective dimension,

$$M \cong \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \otimes^{\mathrm{L}}_A D$$

by Foxby equivalence (Theorem 5.4), so it follows that

$$0 = \sup M = \sup(\mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \otimes^{\mathrm{L}}_A D) = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) - \operatorname{depth}_A A \quad (5.4)$$

by Lemma 2.4. Moreover, $\mathrm{R}\underline{\mathrm{Hom}}_A(D, M)$ has finite projective dimension by Foxby equivalence. We can compute

$$\begin{aligned} \operatorname{depth} \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) &= -\sup \mathrm{R}\underline{\mathrm{Hom}}_A(\mathrm{R}\underline{\mathrm{Hom}}_A(D, M), D) \\ &= -\sup \mathrm{R}\underline{\mathrm{Hom}}_A(\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(\mathrm{R}\underline{\mathrm{Hom}}_A(M, D), A), D) \\ &= -\sup(\mathrm{R}\underline{\mathrm{Hom}}_A(A, D) \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\mathrm{Hom}}_A(M, D)) \\ &= -\sup D - \sup \mathrm{R}\underline{\mathrm{Hom}}_A(M, D) \\ &= \operatorname{depth}_A A + \operatorname{depth}_A M, \qquad (5.5) \end{aligned}$$

by Lemma 4.2, Lemma 2.2(1) and Lemma 2.4, so it follows that

$$\operatorname{pd}_A \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) = \operatorname{depth}_A A - \operatorname{depth}_A \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \overset{(5.5)}{=} -\operatorname{depth}_A M. \quad (5.6)$$

On the other hand, similar to (5.2), we have

$$\inf \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \geq -\operatorname{pd}_A \mathrm{R}\underline{\mathrm{Hom}}_A(D, M). \quad (5.7)$$

These imply that

$$\begin{aligned} -\inf \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) &\overset{(5.7)}{\leq} \operatorname{pd}_A \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \overset{(5.6)}{=} -\operatorname{depth}_A M \\ &= -\operatorname{depth}_A A \overset{(5.4)}{=} -\sup \mathrm{R}\underline{\mathrm{Hom}}_A(D, M). \end{aligned}$$

Thus $\inf \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(D, M)$. Let $N = h^d \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \in \operatorname{grmod} A$ where $d = \operatorname{depth}_A A$. Then $N[-d] \cong \mathrm{R}\underline{\mathrm{Hom}}_A(D, M)$ in $\mathcal{D}(A)$. We see

$$\operatorname{pd}_A N = \sup \mathrm{R}\underline{\mathrm{Hom}}_A(N[-d], A) + d = \operatorname{pd}_A \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) + \operatorname{depth}_A A = 0.$$

So N is free. Put $N = \bigoplus_i A(s_i)$. Since

$$\bigoplus_i D(s_i)[-d] \cong (N \otimes^{\mathrm{L}}_A D)[-d] \cong N[-d] \otimes^{\mathrm{L}}_A D \cong \mathrm{R}\underline{\mathrm{Hom}}_A(D, M) \otimes^{\mathrm{L}}_A D \cong M \in \mathcal{A}(A),$$

we obtain $D \in \mathcal{A}(A)$ by Lemma 3.6. Hence A is AS-Gorenstein by Theorem 5.10. □

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