SEMISIMPLE HOPF ALGEBRAS OF DIMENSION 12

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Introduction

Recently the project of classifying semisimple Hopf algebras over an algebraically closed field is in progress. Under some assumptions on characteristic $\operatorname{ch} k$ of base field k, various classification results are obtained by some authors as listed below.

	dimension	$\operatorname{ch} k$
[LR]	odd number ≤ 19	k is arbitrary field.
[Z]	p	$\operatorname{ch} k = 0$
[M1]	6	$\operatorname{ch} k \neq 2, 3$
[M1]	8	$\operatorname{ch} k \neq 2$
[M2]	$2p$	$\operatorname{ch} k = 0$
[M3]	p^3 $(p \neq 2)$	$\operatorname{ch} k = 0$
[M4]	p^2	$\operatorname{ch} k = 0$
[KM]	$3p$ $(p \equiv 2 \bmod 3)$	$\operatorname{ch} k = 0$
[M5]	18	$\operatorname{ch} k = 0$
[M6]	21	$\operatorname{ch} k = 0$

Here p is a prime number. In fact, any Hopf algebras of prime dimension are necessarily semisimple in the case $\operatorname{ch} k = 0$, and classification of such Hopf algebras is completed.

In this note we classify 12-dimensional semisimple Hopf algebras over an algebraically closed field k whose characteristic $\operatorname{ch} k \neq 2, 3$.

1. For a finite group G, there are two Hopf algebras associated with G.

Example. (1) The group algebra kG is a Hopf algebra with the coproduct

$$\Delta : kG \to kG \otimes kG, \quad G \ni g \mapsto g \otimes g.$$

(2) Denote by k^G the algebra of functions of G. Thus k^G is the set of all functions from G to k, and has the product such that

$$(f \cdot f')(g) = f(g) f'(g) \qquad (f, f' \in k^G, g \in G).$$

For each $g \in G$, define $e_g \in k^G$ by

$$e_g : G \to k, \quad h \mapsto \delta_{h,g} = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{otherwise.} \end{cases}$$

Then $k^G = \prod_{g \in G} ke_g$ is isomorphic as an algebra to $k \times \cdots \times k$ ($|G|$ times), which is commutative and semisimple. Further k^G becomes a Hopf algebra with the coproduct determined by

$$\Delta : k^G \to k^G \otimes k^G, \quad e_g \mapsto \sum_{h \in G} e_h \otimes e_{h^{-1}g}.$$

Let A be a finite-dimensional Hopf algebra. Its dual space $A^* = \mathrm{Hom}_k(A, k)$ has the Hopf algebra structure induced by that of A. Note that A is commutative if and only if A^* is cocommutative, and that A is semisimple if and only if A^* is cosemisimple. Since $k^G = (kG)^*$, kG is cocommutative and cosemisimple. Moreover, for a finite group G, Maschke's theorem says that the characteristic $\mathrm{ch}\, k$ of k does not divide $|G|$ if and only if both kG and k^G are semisimple and cosemisimple.

We call these Hopf algebras kG and k^G *trivial.*

As easily seen, for a finite-dimensional semisimple cosemisiple Hopf algebra A over k, it follows that A is trivial if and only if A is commutative or cocommutative. Hence, in order to classify semisimple cosemisimple Hopf algebras over k it sufficies to classify ones that are neither commutative nor cocommutative.

2. For the classification, the following result is crucial.

Proposition ([LR]). *Let A be a Hopf algebra of dimension 12 over k whose characteristic $\mathrm{ch}\, k \neq 2, 3$. Then A is semisimple if and only if A is cosemi-aimple.*

Note that , if $\mathrm{ch}\, k \neq 2, 3$ then both kG and k^G are semisimple and cosemisimple.

Theorem. *Let k be an algebraically closed field whose characteristic $\mathrm{ch}\, k \neq 2, 3$. Then there exist only two (up to isomorphism) non-trivial semisimple (necessarily cosemisimple) Hopf algebras (A_+, A_-) of dimension 12 over k.*

3. In this section we shall construct the Hopf algebras A_+ and A_-. Suppose that $\mathrm{ch}\, k \neq 2, 3$. Denote by $\mathfrak{S}_3$ the symmetric group of degree 3. The signature map sgn : $\mathfrak{S}_3 \to k$ is defined by

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is a even permutation,} \\ 0 & \text{if } \sigma \text{ is a odd permutation,} \end{cases}$$

for $\sigma \in \mathfrak{S}_3$. Let $\tau = (1\ 2)$. Denote by $\iota^* : k^{\mathfrak{S}_3} \to k^{\mathfrak{S}_3}$ the dual map of the inner automorphism $\iota = \operatorname{inner}(\tau)$ of $\mathfrak{S}_3$. Thus for any $\sigma \in \mathfrak{S}_3$,

$$\iota^*(e_\sigma) = e_{\tau\sigma\tau}.$$

Definition. Denote by A_+ (resp. A_-) the $k^{\mathfrak{S}_3}$-ring generated by z with relations

$$z^2 = 1 \text{ (resp. sgn)}, \qquad zc = \iota^*(c)z \quad (c \in k^{\mathfrak{S}_3}).$$

Note that as algebra $A_+(\cong A_-)$ is semisimple. Further we can make both A_+ and A_- into Hopf algebras by defining

$$\Delta(z) = z \otimes z, \qquad \Delta(c) = \Delta_{k^{\mathfrak{S}_3}}(c) \quad (c \in k^{\mathfrak{S}_3}).$$

A_+ and A_- are mutually non-isomorphic non-trivial semisimple Hopf algebras of dimension 12.

Remark. Suppose that $\operatorname{ch} k = 2$ or 3. As noted before, kG is not semsisimple for any group G of order 12. We point out that, if $\operatorname{ch} k = 2$, then $A_+ = A_-$ is not semisimple from the definition. By [LR, Prop.1.3] 12-dimensional semisimple Hopf algebras over k cannot be cosemisimple. So there may exist non-trivial semisimple Hopf algebras of dimension 12 over k other than A_+ and A_-. In the case $\operatorname{ch} k = 2$, consider the $k^{C_2 \times C_2}$-ring A generated by g with the relations

$$g^3 = 1, \qquad g e_{ij} = e_{j-i,i} g,$$

where $C_2 \times C_2$ is the product of two cyclic group of order 2 with generators a, b, respectively, and e_{ij} is the dual basis of $a^i b^j \in C_2 \times C_2$. Then A becomes a Hopf algebra with the coproduct determined by

$$\Delta(g) = g \otimes g, \qquad \Delta(c) = \Delta_{k^{C_2 \times C_2}}(c) \quad (c \in k^{C_2 \times C_2}).$$

One sees that A is a non-trivial semisimple Hopf algebras of dimension 12.

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