On the Association Schemes of Quadratic Forms

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Abstract

The association scheme of quadratic forms (or the quadratic forms scheme, for short) is one of the known P- and Q-polynomial schemes and its first eigenmatrix is represented by using the Askey-Wilson polynomials. We consider two fission schemes of the quadratic forms scheme in characteristic 2 and describe the first eigenmatrix of one of these fission schemes, and compute some intersection numbers of the other.

1 Introduction

Properties of quadratic forms defined on a vector space V over F_q are quite different depending on whether q is even or odd. If q is odd, there is a one-to-one correspondence between the set of quadratic forms and the set of symmetric bilinear forms. Hence the study of quadratic forms is reduced to that of symmetric bilinear forms. But if q is even, this is no longer the case. Let f be a quadratic form on V and let B_f be the symmetric bilinear form associated to f. Then we have

$$B_f(x,x) = 2f(x) = 0 \quad \text{for any } x \in V,$$

so that B_f is alternating. Given an alternating bilinear form B, there are many quadratic forms associated to B.

This is reflected on the structure of quadratic forms scheme, which is one of the known P- and Q-polynomial schemes. Let X be the set of quadratic forms on V. By defining some relations we can construct a P-and Q-polynomial scheme, which is called the quadratic forms scheme. The quadratic forms scheme has the same parameters as the alternating forms scheme with one dimension larger, but they are not isomorphic (see [4]).

In this paper we consider two fission schemes of the quadratic forms scheme in characteristic 2 and describe the first eigenmatrix of one of these fission schemes, and compute some intersection numbers of the other.

2 Elementary properties of quadratic forms

Let V be a vector space over F_q with $dimV = n$, where q is a prime power.

Definition 2.1 *A function f from V to F_q is called a quadratic form on V if the following conditions hold:*
(i) $f(au) = a^2 f(u)$ *for any* $u \in V,\ a \in F_q$
(ii) $f(u+v) = f(u) + f(v) + B_f(u,v)$ *for any* $u, v \in V$,
where B_f is a symmetric bilinear form on V.

Let B be a symmetric bilinear form on V. Then the radical of B is defined by the following:

$$Rad(B) := \{u \in V | B(u,v) = 0 \quad \text{for any } v \in V\}.$$

Next we define the radical of a quadratic form f to be

$$Radf := RadB_f \cap f^{-1}(0)$$

and the rank of f to be

$$rank(f) := dimV - dim(Radf).$$

Definition 2.2 *Let f be a quadratic form on V. f is called nondegenerate if $Radf = 0$.*

Proposition 2.1 *Let q be even. Let f be a nondegenerate quadratic form on V. Then there is a basis $\{e_1, \cdots, e_n\}$ of V such that*

$$f(u) = \sum_{i=1}^{j} x_{2i-1}x_{2i} + \begin{cases} x_{2j+1}x_{2j+2}, & \text{if } dimV = 2j+2 \cdots (I) \\ x_{2j+1}x_{2j+2} + x_{2j+1}^2 + \xi x_{2j+2}^2, & \text{if } dimV = 2j+2 \cdots (II) \\ x_{2j+1}^2, & \text{if } dimV = 2j+1 \end{cases}$$

where $u = \sum_{i=1}^{n} x_i e_i$, and ξ is an element of F_q for which $t^2+t+\xi$ is irreducible in $F_q[t]$.

(Proof)(see [3], p.59) $\square$

Let $GL(V)$ be the general linear group on V. Let X be the set of quadratic forms on V. If we define the action of $GL(V)$ on X by

$$GL(V) \ni T,\ X \ni f, \quad Tf(v) = f \circ T^{-1}(v) \quad \text{for any } v \in V,$$

then X is partitioned into the following orbits by the above proposition: $\{f \in V | rank(f) = 2l\}$ $(l = 1, 2, \cdots)$ are divided into type (I) and type (II), which are respectively called hyperbolic and elliptic type, while

$$\{f \in V | rank(f) = 2l - 1\} \ (l = 1, 2, \cdots)$$

are themselves orbits.

3 Preliminaries on association schemes

Definition 3.1 (Association scheme) *Let X be a finite set. Let R_i $(i = 0, \ldots, d)$ be subsets of $X \times X$. The configuration $(X, \{R_i\}_{i=0,\ldots,d})$ is called an association scheme if it satisfies the following conditions:*
(i)$R_0 = \{(x, x) | x \in X\}$
(ii)$R_0 \cup R_1 \cup \ldots \cup R_d = X \times X$.

$$\text{For any } i,\ j\ (i \neq j),\ R_i \cap R_j = \phi.$$

(iii)For any i, there is some $i' \in \{0, 1, \ldots, d\}$ such that

$$R_i^t := \{(y, x) | (x, y) \in R_i\} = R_{i'}$$

(iv)$|\{z \in X | (x, z) \in R_i, (z, y) \in R_j\}|$ *is constant whenever $(x, y) \in R_k$, which is denoted by p_{ij}^k, called the intersection number.*

Definition 3.2 (Association scheme) *Let X be a finite set. Let $R_i (i = 0, \ldots, d)$ be subsets of $X \times X$. We define the adjacency matrix of R_i as follows:*

$$(A_i)_{xy} := \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

The configuration $(X, \{R_i\}_{i=0,\ldots,d})$ is called an association scheme if it satisfies the following conditions:

(i) $A_0 = I$, *where* I *is the identity matrix.*
(ii) $A_0 + \ldots + A_d = J$, *where* J *is the all 1 matrix.*
(iii)For any i, *there is some* $i' \in \{0, 1, \ldots, d\}$ *such that*

$$^tA_i = A_{i'}.$$

(iv)For any $i,\ j$,

$$A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$

Note that Definition 3.1 and Definition 3.2 are equivalent.

Let $(X, \{R_i\}_{i=0,\ldots,d})$ be an association scheme. $(X, \{R_i\}_{i=0,\ldots,d})$ is called commutative if $A_iA_j = A_jA_i$ for any $i,\ j$, and symmetric if $A_i^t = A_i$ for any i. Suppose that $(X, \{R_i\}_{i=0,\ldots,d})$ is a commutative association scheme. Then each A_i is diagonalizable by a unitary matrix and $\{A_0, A_1, \cdots, A_d\}$ are pairwise commutative. Hence $\{A_0, A_1, \cdots, A_d\}$ are simultaneously diagonalizable by a unitary matrix. Let W be a vector space over the complex field with $dimW = |X|$. By the above fact W can be written as the orthogonal sum of maximal common eigenspaces, say

$$W = W_0 \perp W_1 \perp \cdots \perp W_r.$$

We may assume $W_0 :=< (1, \cdots, 1) >_C$ since $< (1, \cdots, 1) >_C$ is a maximal common eigenspace. Also we see $r = d$. (see [1], p.58) Now we consider $< A_0, \cdots, A_d >_C$, which is called the Bose-Mesner algebra and is a commutative ring. We know that there is a unique basis $\{E_0, \cdots, E_d\}$ consisting of primitive idempotents. (see [1], p.58)

Definition 3.3 *Let* $(X, \{R_i\}_{i=0,\ldots,d})$ *be an association scheme.*
The first eigenmatrix of $(X, \{R_i\}_{i=0,\ldots,d})$ *is the transformation matrix of the basis* $\{E_0, \cdots, E_d\}$ *to* $\{A_0, A_1, \cdots, A_d\}$. *i.e.*

$$(A_0, A_1, \cdots, A_d) = P(E_0, \cdots, E_d)$$

where P *is in* $M_{d+1}(C)$ *and written as:*

$$P = \begin{pmatrix} p_0(0) & p_1(0) & \cdots & p_d(0) \\ p_0(1) & & & \\ \cdots & & & \\ p_0(d) & \cdots & & p_d(d) \end{pmatrix}$$

where $p_i(j)$ is the eigenvalue of A_i on W_j.

Definition 3.4 *Let $(X, \{R_i\}_{i=0,\ldots,d})$ be a symmetric association scheme. The association scheme $(X, \{R_i\}_{i=0,\ldots,d})$ is called a P-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$ if*

$$A_i = v_i(A_1) \quad (i = 0, 1, \cdots, d)$$

for some polynomials $v_i(x)$ of degree i.
$(X, \{R_i\}_{i=0,\ldots,d})$ is called a Q-polynomial scheme with respect to the ordering $E_0, E_1, \cdots, E_d$ if

$$E_i = v_i^*(E_1) \quad (i = 0, 1, \ldots, d)$$

for some polynomials $v_i^*(x)$ of degree i under the Hadamard product.
The association scheme $(X, \{R_i\}_{i=0,\ldots,d})$ is called a P- and Q-polynomial scheme if it is both P-polynomial and Q-polynomial scheme.

4 Quadratic forms schemes

Let V be a vector space over F_q with $dimV = n$, where q is a power of 2.

Lemma 4.1 *Let X be the set of quadratic forms on V. Define the relations R_i on X as follows:*

$$R_i \ni (f, g) \quad \Longleftrightarrow rank(f - g) = i$$

Then $\mathcal{X} = (X, \{R_i\}_{i=0,\ldots,n})$ is a symmetric association scheme.

(Proof) Define $\sigma_g, \pi_T, \kappa_v : X \to X$ for any $g \in X,\ T \in GL(V),\ v \in V$ by the following:

$$\sigma_g(f) = f + g$$
$$\pi_T(f) = f \circ T^{-1}$$
$$\kappa_v(f) = f + B_f(\cdot, v)^2$$

for any $f \in X$.
Let $G =< \sigma_g, \pi_T, \kappa_v | g \in X,\ T \in GL(V),\ v \in V >$. Then G acts on X transitively. The orbits on $X \times X$ of $GL(V)$ are exactly $\{R_i\}_{i=0,\ldots,n}$. Since we

know that if a permutation group of X acts transitively then the relations on X defined by the orbits of G on $X \times X$ satisfy the definition of association schemes (see [1], p.53), the above lemma holds. $\square$

Proposition 4.1 *Let $\mathcal{Y} = (X, \{\Delta_i\}_{i=0,\dots,d})$, where $\Delta_i = R_{2i-1} \cup R_{2i}$ and $d = [\frac{n+1}{2}]$. Then $\mathcal{Y}$ is a P- and Q-polynomial scheme. Moreover the first eigenmatrix of $\mathcal{Y}$ is represented by Askey-Wilson polynomials.*
More precisely, let $P_{\mathcal{Y}}$ be the first eigenmatrix of $\mathcal{Y}$.

$$P_{\mathcal{Y}} = \begin{pmatrix} 1 & \theta_0 & \cdots & & v_d(0) \\ 1 & \theta_1 & \cdots & & \\ \cdots & \cdots & v_i(\theta_j) & \cdots & \\ 1 & \theta_d & \cdots & & v_d(d) \end{pmatrix}$$

Then $v_i(\theta_j) = k_i u_i(\theta_j)$,
where

$$u_i(\theta_j) = {}_3\Phi_2 \begin{pmatrix} q^{-2i}, & q^{-2j}, & 0, & ; q^2, & q^2 \\ q^{-n-1}, & q^{-n}, & & & \end{pmatrix}$$

with

$$\theta_j = \theta_0 + (q^{-2j} - 1)h, \quad \theta_0 = \frac{(q^n - 1)(q^{n+1} - 1)}{q^2 - 1}, \quad h = \frac{q^{2n+1}}{q^2 - 1}$$

and $k_i := p^0_{ii}$, Φ is defined by the following:

$${}_{r+1}\Phi_r \begin{pmatrix} a_1, & \cdots, & a_{r+1}, & ; p, & x \\ b_1, & \cdots, & b_{r+1}, & & \end{pmatrix} = \sum_{t=0}^{\infty} \frac{(a_1; p)_t \cdots (a_{r+1}; p)_t x^t}{(b_1; p)_t \cdots (b_r; p)_t (p; p)_t}$$

where

$$(a; p)_t = \begin{cases} (1 - a) \cdots (1 - ap^{t-1}) & (t = 1, 2, \cdots) \\ 1 & t = 0 \end{cases}$$

(Proof)(see [1], p.308, [2], p.290) $\square$

5 Computation of $P_{\mathcal{X}}$

Let $\{A_i\}_{0\leq i\leq n}$ be the adjacency matrices of $\mathcal{X}$. Let $\{M_i\}_{0\leq i\leq d}$ be the adjacency matrices of $\mathcal{Y}$. Then the following lemmas hold.

Lemma 5.1 *The following equality holds.*

$$A_1 A_{2j} = (q^{2j}-1)A_{2j} + q^{2j}A_{2j+1} \quad (j=1,2,\cdots)$$

(Proof) For a fixed $(0,\delta)\in R_k$, we will compute

$$p^k_{1\ 2j} = |\{\gamma\in X | rank(\gamma)=1,\ rank(\gamma+\delta)=2j\}|.$$

Note that for a quadratic form f, $rank(B_f)=2j$ if and only if $rank(f)=2j$ or $2j+1$. In this case

$rank(\gamma+\delta)=2j \Rightarrow rank(B_{\delta+\gamma})=2j$

$\Longleftrightarrow rank(B_\delta + B_\gamma)=2j$

$\Longleftrightarrow rank(B_\delta)=2j$

$\Longleftrightarrow rank(\delta)=2j$ or $2j+1$

Hence we have $p^k_{1\ 2j}=0$ for $k\neq 2j$ or $2j+1$. Suppose $rank(\delta)=2j$. Then $\gamma|_{Rad\delta}=0$ if and only if $rank(\gamma+\delta)=2j$. There are $q^{2j}-1$ such γ with $rank(\gamma)=1$, so $p^{2j}_{1\ 2j}=q^{2j}-1$. Similarly we see $p^{2j+1}_{1\ 2j}=q^{2j}$. □

Lemma 5.2 *The eigenvalues of A_1 are $-1,\ q^n-1$.*

(Proof) This is clear, since $A_1^2=(q^n-1)A_0+(q^n-2)A_1$ by a similar calculation as Lemma 5.1. □

Let W be a vector space over the complex field with $dimW=|X|$. Recall from section 3 that W is decomposed into an orthogonal sum of the maximal common eigenspaces of $\{A_0,\cdots,A_n\}$, say

$$W = W_0 \perp \cdots \perp W_n.$$

Similarly W is decomposed into an orthogonal sum of the maximal eigenspaces of $\{M_0,\cdots,M_d\}$, say

$$W = U_0 \perp \cdots \perp U_d.$$

We may assume that the eigenvalue of M_1 on U_i is θ_i, where θ_i $(i=0,1,\cdots,d)$ is defined in Proposition 4.1.

Proposition 5.1 *If n is even, then for any $i \in \{1, \cdots, d\}$ there is a unique 2-subset $\{j_1(i), j_2(i)\}$ of $\{1, \cdots, n\}$ such that $U_i = W_{j_1(i)} \perp W_{j_2(i)}$ where the eigenvalue of A_1 on $W_{j_1(i)}$ is -1, that of A_1 on $W_{j_2(i)}$ is $q^n - 1$.*

If n is odd, then for any $i \in \{1, \cdots, d-1\}$ there is the unique 2-subset $\{j_1(i), j_2(i)\}$ of $\{1, \cdots, n\}$ such that $U_i = W_{j_1(i)} \perp W_{j_2(i)}$ where the eigenvalue of A_1 on $W_{j_1(i)}$ is -1, that on $W_{j_2(i)}$ is $q^n - 1$. Moreover U_d is a maximal common eigenspace of $\{A_0, \cdots, A_n\}$ and the eigenvalue of A_1 on U_d is -1.

(Proof) Since U_i $(i = 1, \cdots, d)$ is a common eigenspace of $\{M_0, \cdots, M_d\}$, for any $i \in \{1, \cdots, d\}$ we can denote $U_i = W_{l_1(i)} \perp \cdots \perp W_{l_r(i)}$ for some $l_1(i), \cdots, l_r(i) \in \{1, \cdots, n\}$.

First we show $r \leq 2$. Suppose $r \geq 3$. Since the eigenvalues of A_1 are -1 and $q^n - 1$, there exists a pair in $\{W_{l_1(i)}, \cdots, W_{l_r(i)}\}$ such that A_1 has the same eigenvalue on them. We may assume that they are $W_{l_1(i)}, W_{l_2(i)}$ without loss of generality. We can determine the eigenvalue of A_2 on $W_{l_1(i)} \perp W_{l_2(i)}$ since the eigenvalue of $A_1 + A_2$ on $W_{l_1(i)} \perp W_{l_2(i)}$ is only $v_1(\theta_i)$. By Lemma 5.1. we can determine the eigenvalue of A_3 on $W_{l_1(i)} \perp W_{l_2(i)}$. Inductively we can determine the eigenvalue of A_k $(j = 0, \cdots, n)$ on $W_{l_1(i)} \perp W_{l_2(i)}$. This contradicts to the maximality of $W_{l_1(i)}$. Hence we have $r \leq 2$.

If n is even then we have $r = 2$ for any i, since $d = [\frac{n+1}{2}]$. Interchaging $l_1(i)$ and $l_2(i)$ if necessary, we may assume that the eigenvalue of A_1 on $W_{l_1(i)}$ is -1, that on $W_{l_2(i)}$ is $q^n - 1$.

If n is odd then there exists the only element of $\{U_1, \cdots, U_d\}$ such that the eigenvalue of A_1 is either -1 or $q^n - 1$. Moreover for any other element there is the unique 2-subset $\{j_1(i), j_2(i)\}$ of $\{1, \cdots, n\}$ such that $U_i = W_{j_1(i)} \perp W_{j_2(i)}$ where the eigenvalue of A_1 on $W_{j_1(i)}$ is -1, that on $W_{j_2(i)}$ is $q^n - 1$.

Moreover we can compute $P_{\mathcal{X}}$ explicitly as follows. Let α_k be the eigenvalue of A_k on $W_{j_1(i)}$ Lemma5.1 we obtain the following equalities:

$$\begin{cases} (-1)\alpha_{2j} = (q^{2j} - 1)\alpha_{2j} + q^{2j}\alpha_{2j+1} \\ \alpha_{2j-1} + \alpha_{2j} = p_j(i) \end{cases}$$

The solution is given by

$$-\alpha_{2j} = \alpha_{2j+1} = -\sum_{k=0}^{j} p_k(i)$$

Let β_k be the eigenvalue of A_k on $W_{j_2(i)}$. Similarly we obtain the following equalities:

$$\begin{cases} (q^n-1)\beta_{2j} = (q^{2j}-1)\beta_{2j} + q^{2j}\beta_{2j+1} \\ \beta_{2j-1} + \beta_{2j} = p_j(i) \end{cases}$$

The solution is given by

$$\beta_{2j+1} = -\sum_{k=0}^{j} p_k(i) \prod_{l=0}^{j-k} (1-q^{n-2j+2l})$$

$$\beta_{2j+1} = p_{j+1}(i) - \beta_{2j+2}$$

Now we show that the eigenvalue of A_1 on U_d is -1 if n is odd. Suppose not. Since $n = 2d-1$, the eigenvalue of A_n on W_{d_2} is

$$\beta_n(d) = -\sum_{k=0}^{d-1} p_k(d) \prod_{l=0}^{d-1-k} (1-q^{2d-1-2(d-1)+2l})$$

$$= -\sum_{k=0}^{d-1} p_k(d)(q;q^2)_{d-k}.$$

On the other hand since $M_d = A_n$, we see $\beta_n(d) = p_d(d)$. We know

$$\theta_d = -\sum_{k=0}^{d-1} p_k(d).$$

Now we observe

$$-\sum_{k=0}^{d-1}(1-(q:q^2)_{d-k})p_k(d).$$

Note that sign of $p_k(d)$ is $(-1)^k$ by the facts that θ_d is the minimum eigenvalue of the P-polynomial scheme $\mathcal{Y}$. This can be seen from the interlacing property of orthogonal polynomials (see [1], p.203).
Note also that sign of $(1-(q:q^2)_{d-k})$ is $(-1)^{k+d}$. So $(1-(q:q^2)_{d-k})p_k(d)$ is positive for all k or negative for all k, which is a contradiction.

The calculation in the proof of Proposition 5.1 gives the following theorem.

Theorem 5.1 *If n is even, the first eigenmatrix of $\mathcal{X}$ is given by*

$$P_{\mathcal{X}} = \begin{pmatrix} 1 & \beta_1(0) & \cdots & & \beta_n(0) \\ 1 & \beta_1(1) & & & \beta_n(1) \\ 1 & \alpha_1(1) & & & \alpha_n(1) \\ \cdots & \cdots & \beta_j(i) & \cdots & \\ \cdots & \cdots & \alpha_j(i) & \cdots & \\ 1 & \alpha_1(d) & & \cdots & \alpha_n(d) \end{pmatrix},$$

and if n is odd,

$$P_{\mathcal{X}} = \begin{pmatrix} 1 & \beta_1(0) & \cdots & & & \beta_n(0) \\ 1 & \beta_1(1) & & & & \beta_n(1) \\ 1 & \alpha_1(1) & & & & \alpha_n(1) \\ \cdots & \cdots & \beta_j(i) & & \cdots & \\ \cdots & \cdots & \alpha_j(i) & & \cdots & \\ 1 & \beta_1(d-1) & \cdots & \cdots & & \\ 1 & \alpha_1(d-1) & \cdots & \cdots & & \\ 1 & \alpha_1(d) & & & \cdots & \alpha_n(d) \end{pmatrix},$$

where

$$\beta_{2j+1}(i) = -\sum_{k=0}^{j} p_k(i) \prod_{l=0}^{j-k}(1 - q^{n-2j+2l}), \qquad \beta_{2j+2}(i) = p_{j+1}(i) - \beta_{2j+1},$$

$$-\alpha_{2j}(i) = \alpha_{2j+1}(i) = -\sum_{k=0}^{j} p_k(i).$$

6 On the parameter of $\mathcal{Z}$

We recall from *section* 2 that $GL(V)$ acts on X and its orbits are

$$\Lambda_{2j+1} = \{f \in V | rank(f) = 2j+1\} \quad (0 \leq j \leq d-1)$$

$$\Lambda_{2j^+} = \{f \in V | rank(f) = 2j, \ f \text{ is non-twisted}\} \quad (1 \leq j \leq \lfloor \frac{n-1}{2} \rfloor),$$

$$\Lambda_{2j^-} = \{f \in V | rank(f) = 2j,\ f \text{ is twisted}\} \quad (1 \le j \le \lfloor \frac{n-1}{2} \rfloor).$$

We construct an association scheme $\mathcal{Z}$ by using $GL(V)$-orbits. Let X be the set of quadratic forms on V. The relations $R_{2j+1}, R_{2j^+}, R_{2j^-}$ are defined as follows:

$$(f,g) \in R_k \quad \Longleftrightarrow \quad f - g \in \Lambda_k,$$

where

$$k \in \{0, 1, 2^+, 2^-, 3, 4^+, 4^-, \cdots, \}.$$

Then $\mathcal{Z} = (X, \{R_i\}_{i=0,1,2^+,2^-,\cdots})$ is an association scheme. Suppose that q is two. We will compute some of the intersection numbers of $\mathcal{Z}$. In particular we want to know the coefficients in the expression of $A_{2^-}A_{2j+1} = \sum_{k\in\{0,1,2^+,2^-,3,4^+,4^-,\cdots,\}} p^k_{2\ 2j+1} A_k$. This equation can be thought a key to obtain a recurssive equation with respect to the eigenvalues of A_{2j^-}'s.

Now we give some lemmas in order to calculate these coefficients. Fix a basis $\{e_1, \cdots, e_n\}$ of V.

Lemma 6.1 *Let B, B' be bilinear forms on V with $rank(B) = b$ and $rank(B) = c$. Let W be a complement of $Rad(B)$ in V and write $U = Rad(B - B') \cap W$. Then $b - c \le rank(B - B') \le b + c$. Moreover:*
(i)$rank(B - B') = b + c$ if and only if $V = Rad(B) + Rad(B')$
(and $B|_U$ is nondegenerate).
(ii)$rank(B - B') = b - c$ if and only if $Rad(B) \subseteq Rad(B')$ and $dimU = c$ (and $B|_U$ is nondegenerate). In this case B' is uniquely determined by $B'|_U = B|_U$ and $Rad(B') = \{v \in V | B(v,U) = 0\}$.

(Proof) (see [2], p.283, Lemma9.5.5) □

Lemma 6.2 *Let f, g be quadratic forms on V with $rank(f) = b$ and $rank(g) = c$. Then $b - c \le rank(f - g) \le b + c$.*

(Proof) In the case that both b and c are even, since $rank(f) = rank(B_f)$, $rank(g) = rank(B_g)$ it follows from *Lemma* 6.1.
Otherwise we have

$$\begin{aligned} rank(f-g) &\le rank(B_{f-g}) + 1 \\ &\le rank(B_f) + rank(B_g) + 1 \\ &\le b + c \end{aligned}$$

Replacing f by $f+g$, we have $b-c \leq rank(f-g)$. $\square$

By Lemma 6.2 we see that $p^k_{2^-\ 2j+1} = 0$ for any $k \in H$ where $H := \{0, 1, 2^+, 2^-, \cdots, (2j-2)^-, (2j+4)^+, (2j+4)^-, (2j+5), \cdots\}$.

Lemma 6.3 *Given any alternating form B on V with $rankB = 2$, there is a unique element $f \in \Lambda_2^-$ such that $B_f = B$.*

(Proof) Since q is two, it is clear. $\square$

Remark 6.1 *If we represent quadratic forms on V as homogeneous polynomials of degree 2 in $x_1, \cdots, x_n$, any element of Λ_2^- is represented as the form of $\rho^2 + \rho\phi + \phi^2$ where*

$$\rho = \sum_{i=1}^{n} \epsilon_i x_i,\ \phi = \sum_{i=1}^{n} \mu_i x_i \quad (\ \epsilon_i,\ \mu_i \in F_2,\ i = 1, \cdots, n).$$

6.1 Computation of $p^{2j^+}_{2^-\ 2j+1}$

For a fixed $(0,\delta) \in R_{2j^+}$, we want to compute $|\{\gamma \in \Lambda_{2^-} | \gamma + \delta \in \Lambda_{2j+1}.\}|$ Since we know that $\mathcal{Z}$ is an association scheme, we may assume $\delta = \sum_{i=1}^{j} x_{2i-1}x_{2i}$. Then we have

$$Rad(B_\delta) = < e_{2j+1}, \cdots, e_n >_C .$$

We divide into the following cases with respect to ρ and ϕ:

$$(i)\rho(Rad(B_\delta)) = \{0\}, \quad \phi(Rad(B_\delta)) = \{0\}$$

$$(ii)\rho(Rad(B_\delta)) \neq \{0\}, \quad \phi(Rad(B_\delta)) \neq \{0\}$$

$$(iii)\rho(Rad(B_\delta)) = \{0\}, \quad \phi(Rad(B_\delta)) \neq \{0\}$$

Lemma 6.4 *In the case (i) there is no element of $\{\gamma \in \Lambda_{2^-} | \gamma + \delta \in \Lambda_{2j+1}\}$, which holds if $\delta \in \Lambda_{2^+}$.*

(Proof) In this case we have

$$Rad\delta = RadB_\delta \subseteq RadB_\gamma = Rad\gamma.$$

So we have

$$Rad\delta \subseteq \gamma^{-1} \cap \delta^{-1} \cap RadB_\gamma \cap RadB_\delta \subseteq (\gamma+\delta)^{-1} \cap RadB_{\delta+\gamma}.$$

Hence we obtain the following formula:

$$\begin{aligned} rank(\gamma+\delta) &= dimV - dimRad(\gamma+\delta) \\ &= dimV - dim\{Rad(B_{\gamma+\delta}) \cap (\gamma+\delta)^{-1}(0)\} \\ &\leq dimV - dim\{Rad(B_{\gamma+\delta}) \cap Rad(B_\delta)\} \\ &= dimV - dimRadB_\delta \\ &= n-(n-2j) = 2j \end{aligned}$$

This proves the lemma. □

Lemma 6.5 *In the case (ii) we have $\phi(u) = \rho(u)$ for any $u \in Rad(B_\delta)$, which holds if $\delta \in \Lambda_{2^+}$.*

(Proof) Otherwise there is $u \in Rad(B_\delta)$ such that $\phi(u) \neq \rho(u)$. We may assume that $\phi(u) = 1$ and $\rho(u) = 0$. By the hypothesis of (ii) there exists $v \in Rad(B_\delta)$ such that $\rho(v) = 1$ and $u \neq v$. We consider the subspace $U =< u, v >$. Since $U \subset Rad(B_\delta)$ and $\{u, v\}$ is a hyperbolic pair with respect to $B_{\delta+\gamma}$ and $RadB_\gamma + RadB_\delta$, we have $rank(B_{\delta+\gamma}) = 2j+2$ by *Lemma* 6.1. This is a contradiction. □

Using the above lemmas and replacing ρ by $\rho+\phi$ we can reduce the case (ii) to the case (iii). Hence it is enough to consider the only case (iii). By the above lemmas we can denote ρ by $\sum_{i=1}^{2j} \epsilon_i x_i$.

$$\delta+\gamma = \sum_{i=1}^{j} N_i$$

where $N_i = x_{2i-1}x_{2i} + \rho^2 + \rho\phi + \phi^2$
$= \phi^2 + \sum_{i=1}^{j}(\epsilon_{2i-1}x_{2i-1} + \epsilon_{2i}x_{2i})\phi + (\epsilon_{2i-1}x_{2i-1})^2 + (\epsilon_{2i}x_{2i})^2 + x_{2i-1}x_{2i}$

$$N_i = \begin{cases} x_{2i-1}x_{2i} & if(\epsilon_{2i-1},\epsilon_{2i}) = (0,0) \\ x_{2i-1}(x_{2i}+\phi) + x_{2i-1}^2 & if \ \ (\epsilon_{2i-1},\epsilon_{2i}) = (1,0) \\ x_{2i}(x_{2i-1}+\phi) + x_{2i}^2 & if \ \ (\epsilon_{2i-1},\epsilon_{2i}) = (0,1) \\ (x_{2i}+\phi)(x_{2i-1}+\phi) + x_{2i}^2 + x_{2i-1}^2 + \phi^2 & if \ \ (\epsilon_{2i-1},\epsilon_{2i}) = (1,1) \end{cases}$$

It follows from the above formula that the contribution of ϕ^2 appears in the only case of $(\epsilon_{2i-1}, \epsilon_{2i}) = (1,1)$. This contribution is associated to the value of $rank(\delta+\gamma)$, i.e. it hold that $rank(\delta+\gamma) = 2j+1$ if and only if the number of contributions of ϕ^2 is odd.
We count the number the choices of $(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2j})$ for which $|\{i|(\epsilon_{2i-1}, \epsilon_{2i}) = (1,1)\}|$ is even. This number is given by $\sum_{t=0}^{\infty}\binom{j}{2t}(2^2-1)^{j-2t}$.

Lemma 6.6

$$\sum_{t=0}^{\infty}\binom{j}{2t}(2^2-1)^{j-2t} = 2^{2j-1} + 2^{2j}$$

(Proof) Let $X_j = \sum_{t=0}^{\infty}\binom{j}{2t}(2^2-1)^{j-2t}$ and $Y_j = \sum_{t=0}^{\infty}\binom{j}{2t+1}(2^2-1)^{j-2t-1}$.
We have

$$(2^2-1+1)^j = \sum_{s=0}^{\infty}\binom{j}{s}(2^2-1)^{j-2t} = X_j + Y_j$$

and

$$Y_j = \sum_{t=0}^{\infty}(\binom{j-1}{2t} + \binom{j-1}{2t+1})(2^2-1)^{j-2t-1} = X_{j-1} + 3Y_{j-1}.$$

Hence we can obtain the following equations.

$$\begin{cases} X_j + Y_j = 2^{2j} \\ Y_j = X_{j-1} + 3Y_{j-1} \end{cases}$$

We solve the above equations to complete the proof of the lemma. $\square$

By Lemma 6.6 and $\rho \neq 0$ we see that the choice of ρ is $2^{2j-1} + 2^{2j} - 1$ and the choice of ϕ is $2^n - 2^{2j}$. Note that $\gamma = \rho^2 + \rho\phi + \phi^2 = \rho(\rho+\phi) + \phi^2$. Since $(\rho+\phi)(Rad\delta) \neq 0$, this implies that we number the choice of γ exactly twice. Hence we obtain the following proposition.

Proposition 6.1

$$p_{2^-\ 2j+1}^{2j^+} = (2^{2j-1} + 2^{2j} - 1)(2^{n-1} - 2^{2j-1})$$

6.2 Computation of $p_{2^- \ 2j+1}^{2j^-}$

For a fixed $(0,\delta) \in R_{2j^-}$, we compute $|\{\gamma \in \Lambda_{2^-} | \gamma + \delta \in \Lambda_{2j+1}\}|$. We may assume

$$\delta = x_{2j-1}^2 + x_{2j}^2 + x_{2j-1}x_{2j} + \sum_{i=1}^{j-1} x_{2i-1}x_{2i}.$$

By Lemma 6.4, 6.5, we may assume $\gamma = \rho^2 + \rho\phi + \phi^2$ where $\rho = \sum_{i=1}^{2j} \epsilon_i x_i, \quad \phi = \sum_{i=1}^{n} \mu_i x_i \quad (\epsilon_i,\ \mu_i \in F_2,\ i = 1, \cdots, n)$.
We have
$\delta + \gamma = x_{2j-1}^2 + x_{2j}^2 + \sum_{i=1}^{j} x_{2i-1}x_{2i} + \rho^2 + \rho\phi + \phi^2$
$\quad = x_{2j-1}^2 + x_{2j}^2 + \phi^2 + \sum_{i=1}^{j} (\epsilon_i x_{2i-1} + \epsilon_i x_{2i})\phi + (\epsilon_i x_{2i-1})^2 + (\epsilon_i x_{2i})^2 + x_{2i-1}x_{2i}$.
For any $i \in \{1, 2, \cdots, j-1\}$ the contribution of ϕ^2 appear in the only case of $(\epsilon_{2i-1}, \epsilon_{2i}) = (1,1)$.
For $i = j$

$$\begin{cases} x_{2j-1}x_{2j} + x_{2j-1}^2 + x_{2j}^2 & if(\epsilon_{2j-1}, \epsilon_{2j}) = (0,0) \\ x_{2j-1}(x_{2j} + \phi) + x_{2j}^2 & if(\epsilon_{2j-1}, \epsilon_{2j}) = (1,0) \\ x_{2j}(x_{2j-1} + \phi) + x_{2j-1}^2 & if(\epsilon_{2j-1}, \epsilon_{2j}) = (0,1) \\ (x_{2j} + \phi)(x_{2j-1} + \phi) + \phi^2 & if(\epsilon_{2j-1}, \epsilon_{2j}) = (1,1) \end{cases}$$

¿From the above formula we see that the contribution of ϕ^2 appear in the case of $(\epsilon_{2j-1}, \epsilon_{2j}) \neq (0,0)$. Now we consider the choice of $(\epsilon_1, \cdots, \epsilon_{2j-2})$ respectively. If $(\epsilon_{2j-1}, \epsilon_{2j}) = (0,0)$, we can take $(X_{j-1} - 1)$'s. If $(\epsilon_{2j-1}, \epsilon_{2j}) = (1,0), (0,1), (1,1)$, we can take Y_{j-1}'s. Hence These are summed up to

$$3Y_{j-1} + X_{j-1} - 1 = Y_j - 1.$$

We obtain the following proposition.
Proposition 6.2

$$p_{2^- \ 2j+1}^{2j^-} = (2^{2j-1} - 2^{2j} - 1)(2^{n-1} - 2^{2j-1})$$

6.3 Computation of $p_{2^- \ 2j+1}^{2j+2^+}$

¿From Lemma 6.1 we see that if $\gamma \in \{\gamma \in \Lambda_{2^-} | \gamma + \delta \in \Lambda_{2j+1}\}$ then we find a 2-space $U \subseteq W$ which is not isotropic with respect to B_γ; and conversely any such U determines B_γ uniquely. Moreover B_γ determines $\gamma \in \Lambda_{2^-}$ uniquely by *Lemma* 6.3. Note that for given such U $rank(\gamma + \delta)$ is not necessarily $2j + 1$, which depends on the choice of U. i.e.

Lemma 6.7 *Let $\delta \in \Lambda_{2^-} \cup \Lambda_{2^-}$, keeping notations above. If $\delta|_U$ is elliptic, then $(\gamma+\delta)|_U = 0$, which follows that $rank(\gamma+\delta) = 2j$. If $\delta|_U$ is hyperbolic, then $(\gamma+\delta)|_U \neq 0$, which follows that $rank(\gamma+\delta) = 2j+1$.*

(Proof) Suppose that $\delta|_U$ is elliptic. Since $dimU = 2$ and $q = 2$, γ is also elliptic, which implies $(\gamma+\delta)|_U = 0$. So we have $(\delta+\gamma)^{-1} \subseteq U$. Then we have

$$(\gamma+\delta)^{-1} \subseteq RadB_{\delta+\gamma} = RadB_{\delta} \oplus U.$$

Hence we have $rank(\gamma+\delta) = 2j$.

Suppose that $\delta|_U$ is hyperbolic. Similarly we have $(\gamma+\delta)|_U \neq 0$. In this case we have $RadB_{\delta+\gamma} = RadB_{\delta} \oplus U$. Since there exist an element $u \in U$ such that $(\delta+\gamma)(u) = 1$. Hence we have $rank(\delta+\gamma) = 2j+1$. Hence we must find 2-space $U \subseteq W$ which $\delta|_U$ is hyperbolic. $\square$

The following lemmas hold.

Lemma 6.8 *Let δ be a nondegenerate twisted quadratic form on W. Let $\mathbf{\Theta} = \{U \subseteq W |\ \delta|_U$ is nontwisted and $dimU = 2\}$. Let $\mathbf{\Omega}$ be the set of unordered pairs $\{u,v\}$ of W such that $\delta(u) = \delta(v) = 0$ and $B_\delta(u,v) = 1$. Then there is a one-to-one correspondence between $\mathbf{\Theta}$ and $\mathbf{\Omega}$.*
Moreover if $dimW = 2j+2$ then $|\mathbf{\Omega}| = 2^{2j-1}(2^{2j+1}+2^j-1)$.

(Proof) (see [3], p.60) $\square$

We obtain the intersection number by these lemmas.

Proposition 6.3 $p^{2j+2^+}_{2^-\ 2j+1} = 2^{2j-1}(2^{2j+1}+2^j-1)$.

6.4 Computation of $p^{2j+2^-}_{2^-\ 2j+1}$

Note that δ is twisted.

Lemma 6.9 *Let δ be a nondegenerate twisted quadratic form on W. Let $\mathbf{\Theta} = \{U \subseteq W |\ \delta|_U$ is nontwisted and $dimU = 2\}$. Let $\mathbf{\Omega}$ be the set of unordered pairs $\{u,v\}$ of W such that $\delta(u) = \delta(v) = 0$ and $B_\delta(u,v) = 1$. Then there is a one-to-one correspondence between $\mathbf{\Theta}$ and $\mathbf{\Omega}$.*
Moreover if $dimW = 2j+2$ then $|\mathbf{\Omega}| = 2^{2j-1}(2^{2j+1}-2^j-1)$.

(Proof) (see [3], p.60) $\square$

By Lemma 6.7, 6.9 we can compute the intersection number as follows:

Proposition 6.4

$$p^{2j+2^+}_{2^- \ 2j+1} = 2^{2j-1}(2^{2j+1} - 2^j - 1).$$

6.5 Computation of $p^{2j+3}_{2^- \ 2j+1}$

Use Lemma 6.1. In this case whenever $\delta|_U$ is ellptic or hyperbolic, we have $rank(\gamma + \delta) = 2j + 1$. So we must count the number of choices of $U \subseteq W$ such that $B_\delta|_U$ is nonisotropic. i.e.

$$|\{U \subseteq W | dimU = 2\}| - |\{U \subseteq W | dimU = 2 \quad B_\delta|_U \text{ is isotropic}\}|$$

Lemma 6.10 *Let B be an alternating form on W. If B is nondegenerate and $dimW = 2l$, then the number of k-subspace of W is*

$$\begin{bmatrix} 2j \\ k \end{bmatrix} := \frac{(2^{2l} - 1) \cdots (2^{2l-k+1} - 1)}{(2^k - 1) \cdots (2 - 1)},$$

and the number of isotropic k-spaces in W is

$$\begin{bmatrix} j \\ k \end{bmatrix} \Pi_{i=0}^{k-1} (2^{j+1-i-1} + 1).$$

(Proof) (see [2], p.274) $\square$

We can calculate this number as follows.

Proposition 6.5

$$p^{2j+3}_{2^- \ 2j+1} = \frac{2^{2j}(2^{2j+2} - 1)}{3}.$$

6.6 Computation of $p_{2^- \ 2j+1}^{2j+1}$

For a fixed $(0,\delta) \in R_{2j+1}$, we compute $|\{\gamma \in \Lambda_{2^-} | \gamma + \delta \in \Lambda_{2j+1}\}|$.
Fix $\delta = \sum_{i=1}^{j} x_{2i-1}x_{2i} + x_{2j+1}^2$. We may assume that γ has the form

$$\phi^2 + \phi\rho + \rho^2$$

where

$$\rho = \sum_{i=1}^{n} \epsilon_i x_i, \ \phi = \sum_{i=1}^{n} \mu_i x_i \ (\epsilon_i, \ \mu_i \in F_2, \ i = 1, \cdots, n).$$

We consider the following cases:

$$(i) \rho(Rad(B_\delta)) = \{0\}, \quad \phi(Rad(B_\delta)) = \{0\}$$
$$(ii) \rho(Rad(B_\delta)) \neq \{0\}, \quad \phi(Rad(B_\delta)) \neq \{0\}$$
$$(iii) \rho(Rad(B_\delta)) \neq \{0\}, \quad \phi(Rad(B_\delta)) = \{0\}$$

Now we consider the case (i).
Let W be a complement of $Rad(B_\delta)$ in V and write $U = RadB_{\gamma+\delta} \cap W$. By Lemma 6.1 we have

$$2j - 2 \leq rank(B_{\gamma+\delta}) \leq 2j + 2.$$

The assumption of (i) follows $rank(B_{\gamma+\delta}) \leq 2j$, $dimU = 2$, and $RadB_\delta \subseteq RadB_\gamma$. By Lemma 6.7 it hold that $rank(B_{\gamma+\delta}) \neq 2j-2$ if and only if $B_\delta|_U$ is not nondegenerate. Hence the choice of such U is as follows:

$$\begin{bmatrix} j \\ 2 \end{bmatrix} \Pi_{i=0}^{2-1}(2^{j+1-i-1} + 1) = \frac{(2^{2j}-1)(2^{2j-2}-1)}{3}$$

Moreover $U \subseteq W$ determines $\gamma \in \Lambda_{2^-}$ uniquely by the condition $RadB_\gamma = \{v \in V | B_\delta(v, U) = 0\}$ and Lemma 6.1. Since $rank(\gamma + \delta) = 2j+1$ for any such γ, the number of choices of γ in the case (i) is

$$\frac{(2^{2j}-1)(2^{2j-2}-1)}{3}.$$

Next we consider the case (ii). Recall Lemma 6.5. Replacing ρ by ρ we see that the case (ii) is reduced to the case (iii).

Finally we consider the case (iii). In this case we may assume $\phi = \sum_{i=1}^{2j} \mu_i x_i$, $\rho = \rho_1 + \rho_2$ where $\rho_1 = \sum_{i=1}^{2j} \epsilon_i x_i$, $\rho_2 = \sum_{i=2j+1}^{n} \epsilon_i x_i$. We divide the case (iii) into the two cases:

(iii)-1 $\rho_2 \neq x_{2j+1}$
(iii)-2 $\rho_2 = x_{2j}$
In the case (iii)-1 whenever we choose ϕ, ρ_1, ρ_2 we have $rank(\gamma+\delta) = 2j+1$. The number of choices of ϕ is $2^{2j}-1$ and that of ρ is $2^n - 2^{2j+1}$. Note that we count the number of choices of γ exactly twice. Hence we obtain the number of choices of γ is

$$\frac{(2^{2j}-1)(2^n - 2^{2j+1})}{2}.$$

Next we consider the case (iii)-2. Note that the value of $rank(\gamma+\delta)$ depends on the choice of $(\epsilon_1, \cdots, \epsilon_{2j}, \mu_1, \cdots, \mu_{2j})$. We research the contributions of ρ^2 similar to section 6.1. Observe the following equation:

$$\gamma+\delta = \sum_{i=1}^{j} x_{2i-1}x_{2i} + (\epsilon_{2i-1}x_{2i-1} + \epsilon_{2i}x_{2i})\rho + (\epsilon_{2i-1}^2 + \mu_{2i-1}^2)x_{2i-1}^2 + (\epsilon_{2i}^2 + \mu_{2i}^2)x_{2i}^2.$$

The number of choices of $(\epsilon_{2i-1}, \epsilon_{2i}, \mu_{2i-1}, \mu_{2i})$ is 16. The cases of $(\epsilon_{2i-1}, \epsilon_{2i}, \mu_{2i-1}, \mu_{2i})$ which appear the contribution of ρ^2 are

$$(1,1,0,0), (1,0,0,1), (0,1,1,0), (0,1,1,1), (1,0,1,1), (1,1,1,1).$$

Hence It holds that $rank(\gamma+\delta) = 2j+1$ if and only if

$$|\{i|(\epsilon_{2i-1}, \epsilon_{2i}, \mu_{2i-1}, \mu_{2i}) \in S\}|$$

is odd where

$$S := \{(1,1,0,0), (1,0,0,1), (0,1,1,0), (0,1,1,1), (1,0,1,1), (1,1,1,1)\}.$$

So $K_j := \sum_{t=0}^{\infty} \binom{j}{2t+1} 6^{2t+1} 10^{j-2t-1}$ is the number we need.

Lemma 6.11

$$K_j = 2^{4j-1} - 2^{2j-1}$$

(Proof) Let $L_j := \sum_{t=0}^{\infty} \binom{j}{2t} 6^{2t} 10^{j-2t}$. Then we can obtain the following equations similar to *Lemma* 6.6:

$$\begin{cases} L_j + K_j = 2^{4j} \\ K_j = 6L_{j-1} + 10K_{j-1} \end{cases}$$

We solve the above equations to complete the proof of the lemma. $\square$

Since we count the number of choices of γ exactly twice in this case,

$$\frac{2^{4j-1}-2^{2j-1}}{2}=2^{4j-2}-2^{2j-2}$$

is the number we need.
We sum the number in the case (i), (iv)-1, (iv)-2 up to obtain the intersection number.
Proposition 6.6

$$p_{2^-\ 2j+1}^{2j+1}=\frac{(2^{2j}-1)(3\cdot 2^n-2^{2j+2}-2)}{6}.$$

6.7 Computation of $p_{2^-\ 2j+1}^{2j-1}$

For a fixed $(0,\delta)\in R_{2j-1}$, we compute $|\{\gamma\in\Lambda_2^-|\gamma+\delta\in\Lambda_{2j+1}\}|$. Fix $\delta=x_{2j-1}^2+\sum_{i=1}^{j-1}x_{2i-1}x_{2i}$. We may assume that γ has the form $\phi^2+\phi\rho+\rho^2$ where $\rho=\sum_{i=1}^n\epsilon_ix_i,\quad \phi=\sum_{i=1}^n\mu_ix_i\quad(\epsilon_i,\ \mu_i\in F_2,\ i=1,\cdots,n)$. Moreover we have that $\rho(Rad(B_\delta))\neq\{0\},\quad \phi(Rad(B_\delta))\neq\{0\}$ and there exists $i_0>2j$ such that $\epsilon_{i_0}\neq\mu_{i_0}$ by Lemma 6.4.

We divide into the following cases:

$(i)\rho(Rad(\delta))=\{0\},\quad \phi(Rad(\delta))=\{0\}$
$(ii)\rho(Rad(\delta))\neq\{0\},\quad \phi(Rad(\delta))\neq\{0\}$
$(iii)\rho(Rad(\delta))=\{0\},\quad \phi(Rad(\delta))\neq\{0\}$.

In the case (i) there is no possibility.

In the case (ii) the number of choices of ϕ is 2^n-2^{2j-1}, the number of choices of ρ which satisfies the above properties is 2^n-2^{2j}. We note that we number γ exactly 6 times to see that the number we need is

$$\frac{(2^n-2^{2j-1})(2^n-2^{2j})}{6}.$$

In the case (iii) We can denote ρ by $x_{2j-1}^2+\sum_{i=1}^{2j-2}\epsilon_ix_i$. We have

$$\delta+\gamma=\sum_{i=1}^{j-1}(x_{2i-1}x_{2i}+\epsilon_{2i-1}^2x_{2i-1}^2+\epsilon_{2i}^2x_{2i}^2)+\rho\phi+\phi^2,$$

which implies $rank(\delta+\gamma)=2j$. Hence there is no possibity.

We obtain the intersection number as follows:

Proposition 6.7

$$p^{2j-1}_{2^- \ 2j+1} = \frac{(2^n - 2^{2j-1})(2^n - 2^{2j})}{6}.$$

These propositions prove the following theorem.

Theorem 6.1 *For any $j \geq 1$ we have that*

$$\begin{aligned} A_{2^-}A_{2j+1} = {} & \tfrac{1}{6}(2^n - 2^{2j-1})(2^n + 2^{2j})A_{2j-1} + \\ & (2^{2j-1} + 2^{j-1} - 1)(2^{n-1} - 2^{2j-1})A_{2j^+} + \\ & (2^{2j-1} - 2^{j-1} - 1)(2^{n-1} - 2^{2j-1})A_{2j^-} + \\ & \tfrac{1}{6}(2^{2j} - 1)(3 \cdot 2^n - 2^{2j+2} - 2)A_{2j+1} + \\ & 2^{2j-1}(2^{2j+1} + 2^j - 1)A_{2j+2^+} + \\ & 2^{2j-1}(2^{2j+1} - 2^j - 1)A_{2j+2^-} + \\ & \tfrac{1}{3}2^{2j}(2^{2j+2} - 1)A_{2j+3} \end{aligned}$$

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