On the characters of height zero

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F : field
A : $F-$algebra

$$\begin{array}{ccc} A & \xrightarrow{F-alg.\ hom.} & M_n(F) : (F-)\text{representation of } A \text{ of degree } n \\ \| & & \downarrow trace \\ A & \xrightarrow[character\ \chi]{} & F \end{array}$$

G : finite group
p : prime
(K, R, k) : $p-$modular system

$$\begin{cases} R : \text{complete discrete valuation ring} \quad (J(R) = (\pi)) \\ K : \text{quotient field of } R \quad (\text{ch } K = 0) \\ k := R/(\pi) \quad (\text{ch } k = p > 0) \end{cases} \quad s.t. \begin{cases} K = K(\sqrt[|G|]{1}) \\ k : \text{algebraically closed} \end{cases}$$

ν : valuation of K $s.t.$ $\nu(p^a q) = a \quad (p \nmid q)$
$\mathfrak{o} := R \ or \ k$

$$\begin{array}{ccc} \mathfrak{o}G & \xrightarrow{\mathfrak{o}-alg.\ hom.} & M_n(\mathfrak{o}) \\ \uparrow & & \uparrow \\ G & \xrightarrow{gp\ hom.} & GL_n(\mathfrak{o}) : (\mathfrak{o}-)\text{representation of } G \text{ of degree } n \end{array}$$

$\longleftrightarrow$ finitely generated $\mathfrak{o}-$free (rank $= n$) $\mathfrak{o}G-$module
(: representation module)

$\mathfrak{o}G-$module := finitely generated left $\mathfrak{o}-$free right $\mathfrak{o}G-$module

$$V : RG - \text{module} \Longrightarrow \begin{cases} V^K := K \bigotimes_R V : KG - \text{module} \\ \qquad\qquad (\leftrightarrow \text{ordinary representation}) \\ V/\pi V : kG - \text{module } (\leftrightarrow \text{modular representation}) \end{cases}$$

$$G \longrightarrow GL_n(R) \longrightarrow GL_n(k)$$
$$(x_{ij})_{i,j} \longrightarrow ({x_{ij}}^*)_{i,j} \quad ({x_{ij}}^* = x_{ij} + (\pi))$$

$\mathfrak{o}G = B_1 \bigoplus \cdots \bigoplus B_n$: indecomposable decomposition
(as $(\mathfrak{o}G, \mathfrak{o}G)$ - bimodule) (B_i : block of G)
$1 = e_{B_1} + \cdots + e_{B_n}$ $(e_{B_i} e_{B_j} = \delta_{i,j} e_{B_i})$ (e_{B_i} : block idempotent of B_i)

$\forall\ V \in \text{IND}(\mathfrak{o}G)\ \exists 1\ B_i \in \text{Bl}(G)$ $s.t.$

$$V\ (= Ve_{B_1} \bigoplus \cdots \bigoplus Ve_{B_n}) = Ve_{B_i}$$

then $V(\leftrightarrow \mathbf{X}, \chi)$ belongs to B_i; $V \in \text{IND}(B_i)$

$V \in \mathrm{IND}(\mathfrak{o}G)$;

$$\mathrm{vx}(V) := \min\{H \leq G;\ V | (V_H)^G (= V_H \bigotimes_{\mathfrak{o}H} \mathfrak{o}G)\}$$

vertex of V : p − group (: up to G − conjugate)

$P := \mathrm{vx}(V)$;

$$\exists s(V) \in \mathrm{IND}(\mathfrak{o}P)\ s.t.\ s(V) | V_P\ ,\ \mathrm{vx}(s(V)) = P$$

$s(V)$: (P−) source of V (: up to $N_G(P)$ − conjugate)

$B \in \mathrm{Bl}(G)$;

$$\delta(B) := \max\{\mathrm{vx}(V);\ V \in \mathrm{IND}(B)\}$$

defect group of B : p − group (: up to G − conjugate)

$d(B) := \nu(|\delta(B)|)$

$V \in \mathrm{IND}(B)(\leftrightarrow \theta)$;

$$(0 \leq)\ ht(V) := \nu(\mathrm{rank}_{\mathfrak{o}} V) - \nu(|G|) + d(B) : \text{height of } V$$
$$= \nu(\theta(1)) - \nu(|G|) + d(B) : \text{height of } \theta$$

$$\begin{cases} \mathbf{X}(\text{: representation}) \\ \chi(\text{: character}) \end{cases} : \text{irreducible} \ :\overset{\text{def}}{\Longleftrightarrow}\ V(\text{: representation module}) : \text{irreducible}$$

Theorem. $\forall\ B \in \mathrm{Bl}(G);\ \mathrm{Irr}^0(B) \neq \emptyset$.

Theorem (Murai). The following conditions on $V \in \mathrm{IND}(B)$ are equivalent.

(i) $ht(V) = 0$

(ii) $\mathrm{vx}(V) =_G \delta(B),\ p \nmid \mathrm{rank}_{\mathfrak{o}} s(V)$

Height zero conjecture (Brauer)

$$B \in \mathrm{Bl}(G);\ \mathrm{Irr}(B) = \mathrm{Irr}^0(B) \iff \delta(B) : \text{abelian}$$

- G : p−solvable group $\Longrightarrow$ O.K.
- ($\Longleftarrow$) : $\forall$ quasi-simple group : O.K. $\Longrightarrow$ $\forall$ group : O.K.

$$\begin{array}{llcclccl}
H \leq G & & & & \| \ H \lhd G & & & \\
\mathfrak{o}G & = B_1 \bigoplus \cdots \bigoplus B_n & b^G & (= B) & \| & B & & B : \text{(weakly) regular} \\
 & & \uparrow & & \| & \downarrow & & \\
 & & \text{defined} & & = & \underset{\text{linked}}{\overset{\text{covers}}{=}} & = & \\
 & & \uparrow & & & \downarrow & \| & \\
\mathfrak{o}H & = b_1 \bigoplus \cdots \bigoplus b_m & b & & & b & \| & (b)
\end{array}$$

1. b^G

$$Z(\mathfrak{o}G) := \bigoplus_{C\in \mathrm{Cl}(G)} \mathfrak{o}\widehat{C} \quad (\widehat{C} := \sum_{x\in C} x \in \mathfrak{o}G)$$

$\chi \in \mathrm{Irr}(G)$;

$$\omega_\chi : Z(RG) \to R : \text{central character of } G \text{ associated with } \chi$$

$$\omega_\chi(\widehat{C}) = \frac{\chi(\widehat{C})}{\chi(1)} = |C|\frac{\chi(x)}{\chi(1)} \quad (x \in C)$$

Theorem. $\chi \in \mathrm{Irr}(B),\ \chi' \in \mathrm{Irr}(B')\ (B, B' \in \mathrm{Bl}(G))$;

(i) $B = B' \Longleftrightarrow \omega_\chi(\widehat{C}) \equiv \omega_{\chi'}(\widehat{C})\ (\equiv: \omega_B(\widehat{C}))\ (\mathrm{mod}\ \pi) \qquad (\forall\ C \in \mathrm{Cl}(G))$

(ii) $\{\omega_B; B \in \mathrm{Bl}(G)\}$ gives all the irreducible $k-$representations of $Z(kG)$;

$$\omega_B : Z(kG) = Z(RG)/\pi Z(RG) \to k$$

$H \leq G$;

$$s_H : Z(kG) \to Z(kH) :\ \widehat{C} \mapsto \widehat{C\cap H} : k - \text{linear map}$$

$b \in \mathrm{Bl}(H)$;

$$\omega_b \circ s_H : Z(kG) \to Z(kH) \to k : k - \text{linear map}$$

If this is a k-algebra homomorphism, then

$$\exists 1B \in \mathrm{Bl}(G)\ s.t.\ \omega_b \circ s_H = \omega_B \qquad \cdots\cdots b^G = B,\ b^G \text{ is defined.}$$

Theorem. b^G is defined. $\Longrightarrow \delta(b) \leq_G \delta(b^G), Z(\delta(b^G)) \leq_G Z(\delta(b))$
in particular ; $d(b) = 0 \Longleftrightarrow d(b^G) = 0$

Example. $H \leq G,\ b \in \mathrm{Bl}(H)$;

(i) $\zeta \in \mathrm{Irr}(b)$

$$\omega_\zeta \circ s_H(\widehat{C}) = \frac{\zeta^G(\widehat{C})}{\zeta^G(1)} \quad (\forall\ C \in \mathrm{Cl}(G))$$

If ζ^G is irreducible, then $\omega_\zeta{}^* \circ s_H = \omega_{\zeta^G}{}^*$ is a $k-$algebra homomorphism.
(i.e. b^G is defined, $\zeta^G \in \mathrm{Irr}(b^G)$)

(ii) $C_G(\delta(b)) \leq H \Longrightarrow b^G$ is defind.

(iii) $1_G : G \to \mathfrak{o}^\times : x \mapsto 1_\mathfrak{o}$: irreducible representation ($\leftrightarrow \mathfrak{o}_G$: representation module);

$$\exists\ B_0(G) \in \mathrm{Bl}(G)\ s.t.\ 1_G \in \mathrm{Irr}(B_0(G)) \qquad \cdots\cdots B_0(G) : \text{principal block of } G$$

$$0 \leq ht(1_G) = \nu(1_G(1)) - \nu(|G|) + d(B_0(G))$$
$$= -\nu(|G|) + d(B_0(G)) \leq 0 \qquad \text{i.e. } \delta(B_0(G)) \in \mathrm{Syl}_p(G)$$

The Third Main Theorem (by Okuyama).
If b^G is defined, then

$$b^G = B_0(G) \Longleftrightarrow b = B_0(H)$$

2. **Covers**

In this section H is normal in G;

$$\mathfrak{o}H = \bigoplus b \Longrightarrow (\mathfrak{o}H =)(\mathfrak{o}H)^x = \bigoplus b^x \quad (x \in G) \qquad \text{i.e. } b^x \in \mathrm{Bl}(H)$$

$\zeta \in \mathrm{Irr}(b), x \in G, h \in H;$

$$\zeta^x(h) := \zeta(xhx^{-1})$$

then $\zeta^x \in \mathrm{Irr}(b^x)$

Theorem (Clifford). $\forall\ \chi \in \mathrm{Irr}(G);$

$$\exists\ \zeta \in \mathrm{Irr}(H) \text{ s.t. } \chi_H = e \sum_{x \in \mathrm{T}(\zeta)\backslash G} \zeta^x \qquad (\exists\ e \geq 1,\ \mathrm{T}(\zeta) := \{x \in G;\ \zeta^x = \zeta\})$$

$$\begin{array}{ccc} G & \xrightarrow{\ \triangleright\ } & H \\ B \ni \chi & \longrightarrow & \zeta \in b \\ \Big\| & \longrightarrow & \Big\|_G \\ B' \ni \chi' & \longrightarrow & \zeta' \in b' \end{array} \qquad \text{i.e. } b' = b^x \quad (\exists\ x \in G)$$

$\cdots\cdots B$ covers b.

Theorem (Passmann). $B \in \mathrm{Bl}(G),\ b \in \mathrm{Bl}(H),$
$\mathrm{Cl}_G(H) := \{C \in \mathrm{Cl}(G);\ C \subseteq H\};$

$$B \text{ covers } b. \iff \omega_B(\widehat{C}) \equiv \omega_b(\widehat{C}) \pmod{\pi} \qquad (\forall\ C \in \mathrm{Cl}_G(H))$$

Example. $(1_G)_H = 1_H;$

$$\begin{array}{l} G \cdots B_0(G) \cdots \delta(B_0(G)) \in \mathrm{Syl}_p(G) \\ \qquad\ \big\downarrow \text{covers} \qquad \big\downarrow \\ H \cdots B_0(H) \cdots \delta(B_0(H)) \in \mathrm{Syl}_p(H) \end{array} \qquad \text{then } \delta(B_0(G)) \cap H =_H \delta(B_0(H))$$

Theorem (Fong - Knörr). $b \in \mathrm{Bl}(H), B \in \mathrm{Bl}(G|b),$
$\mathrm{T}(b) := \{x \in G;\ b^x = b\};$

(i) $\delta(B) \leq_G \mathrm{T}(b)$
(ii) $\delta(B) \leq \mathrm{T}(b) \Longrightarrow (\delta(B) \geq)\delta(B) \cap H =_H \delta(b)$

Remark.

$$\begin{array}{c} B_1, B_2, \cdots, B_n \\ \big\downarrow \text{covers} \\ \{b^x\} \end{array} \qquad \text{If G/H is a } p- \text{ group, then} \qquad \begin{array}{c} B \\ \big\downarrow \text{covers} \\ \{b^x\}. \end{array}$$

3. (Weakly) Regular

In this section H is normal in G;
$B \in \mathrm{Bl}(G),\ b \in \mathrm{Bl}(H)\ s.t.\ b^G = B$
then $\omega_b \circ s_H = \omega_B$,

$$(*)\ C \in \mathrm{Cl}(G),\ C \cap H = \emptyset \implies \omega_B(\widehat{C}) \equiv 0 \pmod{\pi}$$

If $B \in \mathrm{Bl}(G)$ satisfies $(*)$, then B is called to be regular (with respect to H).

Theorem. $C_G(\delta(B)) \leq H \Longrightarrow B$ is regular.

Theorem. $B \in \mathrm{Bl}(G),\ b \in \mathrm{Bl}(H)$;

$$B = b^G \iff B \text{ is regular and covers } b.$$

then $\{B \in \mathrm{Bl}(G|b);\ B : \text{regular}\} = \{B\}$

$b \in \mathrm{Bl}(H)$
$B \in \mathrm{Bl}(G|b)$: weakly regular (with respect to H)
$:\overset{\text{def}}{\iff}\ d(B) = \max\{d(B);\ B \in \mathrm{Bl}(G|b)\}$

Theorem. B : regular $\Longrightarrow$ B : weakly regular

Theorem. $H \lhd G,\ b \in \mathrm{Bl}(H)$;

$$b^G \text{is defined.} \iff \begin{cases} \{B \in \mathrm{Bl}(G|b);\ B : \text{weakly regular}\} = \{B\} \\ Z(\delta(B)) \leq H \end{cases}$$

Theorem. $B \in \mathrm{Bl}(G|b)$ (: weakly regular) ,
$\chi \in \mathrm{Irr}^0(B) :\ \chi_H = e \sum_{x \in \mathrm{T}(\zeta)\backslash G} \zeta^x\ (\zeta \in \mathrm{Irr}(b))$ then ;

$$ht(\zeta) = 0\ (,\ e|\mathrm{T}(b) : \mathrm{T}(\zeta)| \not\equiv 0 \pmod{p})$$

4. **Linkage**

In this section ;

$$H \leq G,\ b \in \mathrm{Bl}(H),\ B \in \mathrm{Bl}(G)$$

$$\mathrm{Chr}(G) := \bigoplus_{\chi \in \mathrm{Irr}(G)} R\chi \qquad \mathrm{Chr}(B) := \bigoplus_{\chi \in \mathrm{Irr}(B)} R\chi$$

$$\mathrm{Chr}^0(B) := \{\theta \in \mathrm{Chr}(B);\ ht(\theta) = 0\}$$

B and b are linked.
$:\overset{\text{def}}{\Longleftrightarrow}\ \forall\, \theta \in \mathrm{Chr}^0(B);\ \theta_b \in \mathrm{Chr}^0(b)$
$(\Longleftrightarrow\ \forall\, \theta \in \mathrm{Bch}^0(B);\ \theta_b \in \mathrm{Bch}^0(b))$

Theorem. b^G is defined. Then ;

$$B \text{ and } b \text{ are linked.} \iff b^G = B$$

Theorem. $b^G = B$. Then ;

$$\theta \in \mathrm{Chr}(G);\ ht(\theta_B) = 0 \iff ht(\theta_b) = 0$$

Theorem. B and b are linked and $d(b) = d(B)$. Then ;

$$\xi \in \mathrm{Chr}(b);\ ht(\xi) = 0 \iff ht(\xi^B) = 0$$

Theorem. $H \lhd G$; The following conditions are equivalent.

(i) B and b are linked.
(ii) B is weakly regular and covers b.

Example. $B_0(G)$: principal block of G

- $H \lhd G \implies B_0(G)$ covers $B_0(H)$ (hence $B_0(G)$ and $B_0(H)$ are linked).
- $p \big| |G|,\ H : p'-$ subgroup of $G \implies B_0(H)^G$ is not defined.

Reference

1. M.Murai: *Block induction, normal subgroups and characters of height zero*, Osaka J.Math., **31**(1994),9-25.