GELFAND-KIRILLOV DIMENSION FOR QUANTIZED WEYL ALGEBRAS

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Abstract. We obtain an analogue of Bernstein's inequality for quantized Weyl algebras.

1. Introduction

Let k be a field. For an n-tuple $\bar{q} = (q_1, \cdots, q_n) \in (k^\times)^n$ and $n \times n$ matrix $\Lambda = (\lambda_{ij})$ over k such that $\lambda_{ii} = 1$ and $\lambda_{ij} = \lambda_{ji}^{-1}$ for all i, j, the n-th *quantized Weyl algebra* $A_n^{\bar{q},\Lambda}$ is the k-algebra generated by the elements $x_1, \cdots, x_n, y_1, \cdots, y_n$ with the following relations:

$$
\begin{aligned}
x_i x_j &= q_i \lambda_{ij} x_j x_i, \\
y_i y_j &= \lambda_{ij} y_j y_i, \\
x_i y_j &= \lambda_{ji} y_j x_i, \\
y_i x_j &= q_i^{-1} \lambda_{ji} x_j y_i, \\
x_j y_j - q_j y_j x_j &= 1 + \sum_{l=1}^{j-1} (q_l - 1) y_l x_l, \\
(x_1 y_1 - q_1 y_1 x_1 &= 1),
\end{aligned}
\tag{1.1}
$$

where $1 \leq i < j \leq n$. See [AD, 3.4].

This algebra $A_n^{\bar{q},\Lambda}$, appeared in the work of Maltsiniotis on noncommutative differential calculus [Ma], is regarded as a q-analogue of the Weyl algebra A_n.

Bernstein's inequality says that, if M is a nonzero module over the Weyl algebra A_n, then the Gelfand-Kirillov dimension $\mathrm{GKdim}(M) \geq n$. The purpose of this note is to obtain an analogue of this result for quantized Weyl algebra $A_n^{\bar{q},\Lambda}$. To this end, a simple localization of $A_n^{\bar{q},\Lambda}$ studied in [J] plays a important role.

Throughout this note, let $\bar{q}$, Λ be as above, and suppose that no q_i is a root of unity.

For ring theorical notions including localizations, filtrations and Gelfand-Kirillov dimension, we refer to [McR].

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2. Preliminaries

For $1 \leq i \leq n$, let $z_i = 1 + \sum_{l=1}^{i}(q_l - 1)y_l x_l$. By [J, 2.8] these elements satisfy the following relations:

$$z_j y_i = \begin{cases} y_i z_j & \text{if } j < i, \\ q_i y_i z_j & \text{if } j \geq i, \end{cases} \qquad z_j x_i = \begin{cases} x_i z_j & \text{if } j < i, \\ q_i^{-1} x_i z_j & \text{if } j \geq i, \end{cases} \tag{2.1}$$

$$z_i z_j = z_j z_i.$$

Thus, for each i, the set $\mathcal{Z}_i = \{z_i^s\}_{s \geq 0}$ is an Ore set in $A_n^{\bar{q},\Lambda}$, and the set $\mathcal{Z} = \mathcal{Z}_1 \cdots \mathcal{Z}_n$ is too. We denote by $B_n^{\bar{q},\Lambda}$ the localization of $A_n^{\bar{q},\Lambda}$ at $\mathcal{Z}$.

Proposition 2.2 [J, Thm. 3.2]. *Suppose that no q_i is a root of unity. Then $B_n^{\bar{q},\Lambda}$ is simple. In particular $B_n^{\bar{q},\Lambda}$ has no nonzero finite-dimensional module.*

Let us consider standard filtrations for $A = A_n^{\bar{q},\Lambda}$ and $B = B_n^{\bar{q},\Lambda}$.

Put $V = k + kx_1 + \cdots + kx_n + ky_1 + \cdots + ky_n$. This is a (finite-dimensional) generating subspace of A, that is, $A = \sum_{l \geq 0} V^l$, where $V^0 = k$. Then A has the filtration $\mathcal{B}(A)$ defined by

$$\mathcal{B}_s(A) = \sum_{l=0}^{s} V^l.$$

For the localization B of A, the subspace $W = kx_1 + kx_2 z_1^{-1} + \cdots + kx_n z_{n-1}^{-1} + ky_1 + \cdots + ky_n + kz_1 + \cdots + kz_n + kz_1^{-1} \cdots + kz_n^{-1}$ is a generating subspace. Denote by $\Gamma(B)$ the filtration of B associated with the generating subspace W. Thus

$$\Gamma_s(B) = \sum_{l=0}^{s} W^l.$$

A k-algebra R is called *semi-commutative* if R is generated as a k-algebra generated by elements $r_1, \cdots, r_m$ such that $r_i r_j = \mu_{ij} r_j r_i$ for $1 \leq i, j \leq n$, where $\mu_{ij} \in k^\times$ ([Mc, 3.7]).

Lemma 2.3. *The graded algebra $gr_\Gamma B$ of $B_n^{\bar{q},\Lambda}$ associated with the filtration $\Gamma(B)$ is semi-commutative.*

Proof. This is clear from the relations (1.1), (2.1) and the observation that $x_i z_{i-1}^{-1} y_i - q_i y_i x_i z_{i-1}^{-1} = 1$ for each i. □

By the lemma we can apply [Mc, Thm.3.8] to B, so the following proposition is obtained. Also see [McP, Sect. 5] .

Proposition 2.4. *Let notations be as above.*

(1) *For any finitely generated B-module M, the Gelfand Kirillov dimension* $\mathrm{GKdim}_B(M)$ *is nonnegative integer.*

(2) *For any nonzero finitely generated B-module M, there exists a nonnegative integer $e_B(M) \geq 1$ such that*

$$e_B(M) = e_B(L) + e_B(N)$$

for any exact sequence $0 \to L \to M \to N \to 0$ *of finitely generated* B*-modules with* $\mathrm{GKdim}_B(L) = \mathrm{GKdim}_B(M) = \mathrm{GKdim}_B(N)$.

(3) *For a* B*-module with finite length, the endmorphism ring* $\mathrm{End}_B(M)$ *of* M *is algebraic over* k.

$e_B(M)$ is called the multiplicity of M.

3. Main Results

Theorem 3.1. *Suppose that no* q_i *is a root of unity. Let* M *be a nonzero* $B_n^{\bar{q},\Lambda}$*-module. Then*

$$n \leq \mathrm{GKdim}_{B_n^{\bar{q},\Lambda}}(M) \leq 2n.$$

Proof. We modify the proof of [McR, Prop.5.5] to prove the theorem.

Write $B_n = B_n^{\bar{q},\Lambda}$. Let M be a nonzero B-module. Since $\mathrm{GKdim}(B) = 2n$ by [GL, Prop.3.4], it follows that $\mathrm{GKdim}_B(M) \leq 2n$.

We will show the inequality $n \leq \mathrm{GKdim}_B(M)$ by induction on n. We can assume that M is finitely generated. If $n = 1$, it is clear from Proposition 2.4 (1) and Proposition 2.2. Assume that the inequality holds for $n-1$. Let $\bar{q'} = (q_1, \cdots, q_{n-1})$, Λ' be the subarray $(\lambda_{ij})_{1\leq i,j\leq n-1}$ of Λ. Then $B_{n-1} = B_{n-1}^{\bar{q'},\Lambda'}$ can be regarded as a subalgebra of B_n. If $\mathrm{GKdim}_{B_n}(M) < n$, then $\mathrm{GKdim}_{B_n}(M) \leq n-1$. We claim that M has finite length as a B_{n-1}-module. It is sufficient to show that any finitely generated B_{n-1}-submodule of M has finite length $\leq e_{B_n}(M)$. Let N be a finitely generated nonzero B_{n-1}-submodule of M. By the inductive hypothesis, one sees that $n-1 \leq \mathrm{GKdim}_{B_{n-1}}(N) \leq \mathrm{GKdim}_{B_{n-1}}(M) \leq \mathrm{GKdim}_{B_n}(M) = n-1$, so that $\mathrm{GKdim}_{B_{n-1}}(N) = \mathrm{GKdim}_{B_n}(M) = n-1$. Then it follows from [McP, Prop.5.7] that $e_{B_{n-1}}(N) \leq e_{B_n}(M)$. Using Proposition 2.4(2), one sees that N has finite length $\leq e_{B_{n-1}}(N) \leq e_{B_n}(M)$.

Now, by Proposition 2.4 (3), $\mathrm{End}_{B_{n-1}}(M)$ is algebraic over k. From the relations (2.1), left action by $z_n z_{n-1}^{-1}$ on M is a left B_{n-1}-module endomorphism of M. Moreover, since M is faithful as a B-module, the k-algebra generated by $z_n z_{n-1}^{-1}$ can be regarded as a sublagebra of $\mathrm{End}_{B_{n-1}}(M)$. However it is easy to check that $z_n z_{n-1}^{-1}$ is algebraic independent over k, which is a contradiction. □

Corollary 3.2. *Suppose that no* q_i *is a root of unity. Let* M *be a finitely generated* $A_n^{\bar{q},\Lambda}$*-module. If* M *is not* $\mathcal{Z}$*-torsion, then*

$$n \leq \mathrm{GKdim}_{A_n^{\bar{q},\Lambda}}(M) \leq 2n.$$

Proof. Put $A = A_n^{\bar{q},\Lambda}$, $B = B_n^{\bar{q},\Lambda}$. First, we claim that $\mathrm{GKdim}_A(M) = \mathrm{GKdim}_B(B \otimes_A M)$ for any $\mathcal{Z}$-torsionfree finitely generated nonzero A -module M. We modify the proof of [GL, Lemma 3.3] to prove the claim. Let V be the generating subspace of A described before. It is ovbious that $W = V + kz_1^{-1}$ is a generating subspace of the localization $\mathcal{Z}_1^{-1}A$ of A at $\mathcal{Z}_1$. There exists nonnegative integer t such that $W^m \subset z_1^{-m}V^{mt}$ for each m. Let M_0 be a finite-dimensional generating subspace of the A-module M. Then $W^m M_0 \subset z_1^{-m}V^{mt}M_0$, so that $\dim_k W^m M_0 \leq \dim_k V^{mt} M_0$. Since M is $\mathcal{Z}_1$-torsionfree, we can regard M as an A-submodule of $\mathcal{Z}_1^{-1}M = \mathcal{Z}_1^{-1}A \otimes_A M$ via the map $M \to \mathcal{Z}_1^{-1}M$, $m \mapsto 1 \otimes m$. In particular M_0 is a generating subspace of the $\mathcal{Z}_1^{-1}A$-module $\mathcal{Z}_1^{-1}M$. Thus $\mathrm{GKdim}_{\mathcal{Z}_1^{-1}A}(\mathcal{Z}_1^{-1}M) \leq \mathrm{GKdim}_A(M)$. Clearly $\mathrm{GKdim}_{\mathcal{Z}_1^{-1}A}(\mathcal{Z}_1^{-1}M) \geq \mathrm{GKdim}_A(M)$. Hence $\mathrm{GKdim}_{\mathcal{Z}_1^{-1}A}(\mathcal{Z}_1^{-1}M) = \mathrm{GKdim}_A(M)$. By continuing similar argument, we can prove the claim.

Let M be a non-$\mathcal{Z}$-torsion finitely generated A-module. Denote by $T(M)$ the largest $\mathcal{Z}$-torsion submodule of M. Since $M/T(M)$ is a $\mathcal{Z}$-torsionfree nonzero module, it holds that

$$\mathrm{GKdim}_A(M/T(M)) = \mathrm{GKdim}_{\mathcal{Z}^{-1}A}(\mathcal{Z}^{-1}(M/T(M))) \geq n$$

by Theorem 3.1. It follows from [McR, 8.3.2] that $\mathrm{GKdim}_A(M/T(M)) \leq \mathrm{GKdim}_A(M)$, which implies that $n \leq \mathrm{GKdim}_A(M)$.

The upper bound is clear since $\mathrm{GKdim}(A) = 2n$ (see [GL, Prop.3.4]). □

Remark 3.3. The corollary fails without the condition on a $A_n^{\bar{q},\Lambda}$-module M. In fact, for $1 \leq i \leq n$, there exists a $\mathcal{Z}$-torsion finitely generated $A_n^{\bar{q},\Lambda}$-module M with $\mathrm{GKdim}_{A_n^{\bar{q},\Lambda}}(M) = i$. Put $A = A_n^{\bar{q},\Lambda}$. Let $L = Ay_{i+1} + \cdots + Ay_n + Ax_1 + \cdots + Ax_n$. The A-module $M = A/L$ has the filtration $\mathcal{B}'(M)$ induced by the filtration $\mathcal{B}(A)$ of A. Thus $\mathcal{B}'_s(M) = (\mathcal{B}_s(A) + L)/L$ is isomorphic as a vector space to

$$\bigoplus_{\alpha_1+\cdots+\alpha_i \leq s} ky_1^{\alpha_1} \cdots y_i^{\alpha_i}.$$

Hence $\dim_k \mathcal{B}'_s(M) = \binom{i+s}{i}$. This implies that $\mathrm{GKdim}_A(M) = i$.

Another Bernstein's inequality for quantized Weyl algebras has been considered by Demidov in [D].

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