GELFAND-KIRILLOV DIMENSION FOR QUANTIZED WYEAL ALGEBRAS

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Abstract. We obtain an analogue of Bernstein’s inequality for quantized Weyl algebras.

1. Introduction

Let $k$ be a field. For an $n$-tuple $\bar{q} = (q_1, \cdots, q_n) \in (k^\times)^n$ and $n \times n$ matrix $\Lambda = (\lambda_{ij})$ over $k$ such that $\lambda_{ii} = 1$ and $\lambda_{ij} = \lambda_{ji}^{-1}$ for all $i, j$, the $n$-th quantized Weyl algebra $A_{\bar{q}, \Lambda}^n$ is the $k$-algebra generated by the elements $x_1, \cdots, x_n, y_1, \cdots, y_n$ with the following relations:

$$
\begin{align*}
    x_i x_j &= q_i \lambda_{ij} x_j x_i, \\
    y_i y_j &= \lambda_{ij} y_j y_i, \\
    x_i y_j &= \lambda_{ij} y_j x_i, \\
    y_i x_j &= q_i \lambda_{ji} x_j y_i, \\
    x_j y_j - q_j y_j x_j &= 1 + \sum_{l=1}^{j-1} (q_l - 1) y_l x_l, \\
    (x_1 y_1 - q_1 y_1 x_1 &= 1),
\end{align*}
$$

(1.1)

where $1 \leq i < j \leq n$. See [AD, 3.4].

This algebra $A_{\bar{q}, \Lambda}^n$, appeared in the work of Maltsiniotis on noncommutative differential calculus [Ma], is regarded as a $q$-analogue of the Weyl algebra $A_n$.

Bernstein’s inequality says that, if $M$ is a nonzero module over the Weyl algebra $A_n$, then the Gelfand-Kirillov dimension $\text{GKdim}(M) \geq n$. The purpose of this note is to obtain an analogue of this result for quantized Weyl algebra $A_{\bar{q}, \Lambda}^n$. To this end, a simple localization of $A_{\bar{q}, \Lambda}^n$ studied in [J] plays an important role.

Throughout this note, let $\bar{q}, \Lambda$ be as above, and suppose that no $q_i$ is a root of unity.

For ring theoretical notions including localizations, filtrations and Gelfand-Kirillov dimension, we refer to [McR].

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2. Preliminaries

For $1 \leq i \leq n$, let $z_i = 1 + \sum_{l=1}^{i}(q_l - 1)y_l x_l$. By [J, 2.8] these elements satisfy the following relations:

$$
\begin{align*}
  z_j y_i &= \begin{cases} 
  y_i z_j & \text{if } j < i, \\
  q_i y_i z_j & \text{if } j \geq i,
  \end{cases} \\
  z_j x_i &= \begin{cases} 
  x_i z_j & \text{if } j < i, \\
  q_i^{-1} x_i z_j & \text{if } j \geq i,
  \end{cases}
\end{align*}
$$

Thus, for each $i$, the set $Z_i = \{z_i^s\}_{s \geq 0}$ is an Ore set in $A_n^q$, and the set $Z = Z_1 \cdots Z_n$ is too. We denote by $B_n^q$ the localization of $A_n^q$ at $Z$.

**Proposition 2.2** [J, Thm. 3.2]. Suppose that no $q_i$ is a root of unity. Then $B_n^q$ is simple. In particular $B_n^q$ has no nonzero finite-dimensional module.

Let us consider standard filtrations for $A = A_n^q$ and $B = B_n^q$. Put $V = k + kx_1 + \cdots + kx_n + ky_1 + \cdots + ky_n$. This is a (finite-dimensional) generating subspace of $A$, that is, $A = \sum_{l \geq 0} V^l$, where $V^0 = k$. Then $A$ has the filtration $B(A)$ defined by $B_s(A) = \sum_{l \geq 0} V^l$.

For the localization $B$ of $A$, the subspace $W = kx_1 + kx_2 z_1^{-1} + \cdots + kx_n z_{n-1}^{-1} + ky_1 + \cdots + ky_n + kz_1 + \cdots + kz_n + k^{-1}_1 \cdots + k^{-1}_n$ is a generating subspace. Denote by $\Gamma(B)$ the filtration of $B$ associated with the generating subspace $W$. Thus

$$
\Gamma_s(B) = \sum_{l \geq 0} W^l.
$$

A $k$-algebra $R$ is called semi-commutative if $R$ is generated as a $k$-algebra generated by elements $r_1, \cdots, r_m$ such that $r_i r_j = \mu_{ij} r_j r_i$ for $1 \leq i, j \leq n$, where $\mu_{ij} \in k^\times$ ([Mc, 3.7]).

**Lemma 2.3.** The graded algebra $\text{gr}_B$ of $B_n^q$ associated with the filtration $\Gamma(B)$ is semi-commutative.

**Proof.** This is clear from the relations (1.1), (2.1) and the observation that $x_i z_{i-1}^{-1} y_i - q_i y_i x_i z_{i-1}^{-1} = 1$ for each $i$. \[\square\]

By the lemma we can apply [Mc, Thm.3.8] to $B$, so the following proposition is obtained. Also see [McP, Sect. 5].

**Proposition 2.4.** Let notations be as above.

(1) For any finitely generated $B$-module $M$, the Gelfand Kirillov dimension $\text{GKdim}_B(M)$ is nonnegative integer.

(2) For any nonzero finitely generated $B$-module $M$, there exists a nonnegative integer $e_B(M) \geq 1$ such that

$$
\text{e}_B(M) = \text{e}_B(L) + \text{e}_B(N)
$$
for any exact sequence \( 0 \to L \to M \to N \to 0 \) of finitely generated \( B \)-modules with \( \text{GKdim}_B(L) = \text{GKdim}_B(M) = \text{GKdim}_B(N) \).

(3) For a \( B \)-module with finite length, the endomorphism ring \( \text{End}_B(M) \) of \( M \) is algebraic over \( k \).

\( e_B(M) \) is called the multiplicity of \( M \).

3. Main Results

**Theorem 3.1.** Suppose that no \( q_i \) is a root of unity. Let \( M \) be a nonzero \( B^{(n)}_\Lambda \)-module. Then

\[
 n \leq \text{GKdim}_{B^{(n)}_\Lambda}(M) \leq 2n.
\]

**Proof.** We modify the proof of [McR, Prop.5.5] to prove the theorem.

Write \( B_n = B^{(n)}_\Lambda \). Let \( M \) be a nonzero \( B \)-module. Since \( \text{GKdim}(B) = 2n \) by [GL, Prop.3.4], it follows that \( \text{GKdim}_B(M) \leq 2n \).

We will show the inequality \( n \leq \text{GKdim}_B(M) \) by induction on \( n \). We can assume that \( M \) is finitely generated. If \( n = 1 \), it is clear from Proposition 2.4 (1) and Proposition 2.2. Assume that the inequality holds for \( n - 1 \). Let \( q' = (q_1, \cdots, q_{n-1}) \), \( \Lambda' \) be the subarray \((\lambda_{ij})_{1 \leq i,j \leq n-1}\) of \( \Lambda \). Then \( B_{n-1} = B^{(n-1)}_{\Lambda'} \) can be regarded as a subalgebra of \( B_n \). If \( \text{GKdim}_{B_n}(M) < n \), then \( \text{GKdim}_{B_n}(M) \leq n - 1 \). We claim that \( M \) has finite length as a \( B_{n-1} \)-module. It is sufficient to show that any finitely generated \( B_{n-1} \)-submodule of \( M \) has finite length \( \leq e_{B_n}(M) \). Let \( N \) be a finitely generated nonzero \( B_{n-1} \)-submodule of \( M \). By the inductive hypothesis, one sees that \( n - 1 \leq \text{GKdim}_{B_{n-1}}(N) \leq \text{GKdim}_{B_{n-1}}(M) \leq \text{GKdim}_{B_n}(M) = n - 1 \), so that \( \text{GKdim}_{B_{n-1}}(N) = \text{GKdim}_{B_n}(M) = n - 1 \). Then it follows from [McP, Prop.5.7] that \( e_{B_{n-1}}(N) \leq e_{B_n}(M) \). Using Proposition 2.4(2), one sees that \( N \) has finite length \( \leq e_{B_{n-1}}(N) \leq e_{B_n}(M) \).

Now, by Proposition 2.4 (3), \( \text{End}_{B_{n-1}}(M) \) is algebraic over \( k \). From the relations (2.1), left action by \( z_nz_{n-1}^{-1} \) on \( M \) is a left \( B_{n-1} \)-module endomorphism of \( M \). Moreover, since \( M \) is faithful as a \( B \)-module, the \( k \)-algebra generated by \( z_nz_{n-1}^{-1} \) can be regarded as a subalgebra of \( \text{End}_{B_{n-1}}(M) \). However it is easy to check that \( z_nz_{n-1}^{-1} \) is algebraically independent over \( k \), which is a contradiction. \( \square \)

**Corollary 3.2.** Suppose that no \( q_i \) is a root of unity. Let \( M \) be a finitely generated \( A^{(n)}_\Lambda \)-module. If \( M \) is not \( \mathcal{Z} \)-torsionfree, then

\[
 n \leq \text{GKdim}_{A^{(n)}_\Lambda}(M) \leq 2n.
\]

**Proof.** Put \( A = A^{(n)}_\Lambda \), \( B = B^{(n)}_\Lambda \). First, we claim that \( \text{GKdim}_A(M) = \text{GKdim}_B(B \otimes_A M) \) for any \( \mathcal{Z} \)-torsionfree finitely generated nonzero \( A \)-module \( M \). We modify the proof of [GL, Lemma 3.3] to prove the claim. Let \( V \) be the generating subspace of \( A \) described before. It is obvious that \( W = V + kz_1^{-1} \) is a generating subspace of the localization \( \mathcal{Z}_1^{-1}A \) of \( A \) at \( \mathcal{Z}_1 \). There exists nonnegative integer \( t \) such that \( W^m \subset z_1^{-m}V^m \) for each \( m \). Let \( M_0 \) be a finite-dimensional generating subspace of the \( A \)-module \( M \). Then \( W^mM_0 \subset z_1^{-m}V^mM_0 \), so that \( \text{dim}_W W^mM_0 \leq \text{dim}_V V^mM_0 \). Since \( M \) is \( \mathcal{Z}_1 \)-torsionfree, we can regard \( M \) as an \( A \)-submodule of \( \mathcal{Z}_1^{-1}M = \mathcal{Z}_1^{-1}A \otimes_A M \) via the map \( M \to \mathcal{Z}_1^{-1}M, m \mapsto 1 \otimes m \). In particular \( M_0 \) is a generating subspace...
of the $Z^{-1}_1A$-module $Z^{-1}_1M$. Thus $\text{GKdim}_{Z^{-1}_1A}(Z^{-1}_1M) \leq \text{GKdim}_A(M)$. Clearly $\text{GKdim}_{Z^{-1}_1A}(Z^{-1}_1M) \geq \text{GKdim}_A(M)$. Hence $\text{GKdim}_{Z^{-1}_1A}(Z^{-1}_1M) = \text{GKdim}_A(M)$. By continuing similar argument, we can prove the claim.

Let $M$ be a non-$Z$-torsion finitely generated $A$-module. Denote by $T(M)$ the largest $Z$-torsion submodule of $M$. Since $M/T(M)$ is a $Z$-torsionfree nonzero module, it holds that

$$\text{GKdim}_A(M/T(M)) = \text{GKdim}_{Z^{-1}_1A}(Z^{-1}(M/T(M))) = n$$

by Theorem 3.1. It follows from [McR, 8.3.2] that $\text{GKdim}_A(M/T(M)) \leq \text{GKdim}_A(M)$, which implies that $n \leq \text{GKdim}_A(M)$.

The upper bound is clear since $\text{GKdim}(A) = 2n$ (see [GL, Prop.3.4]). □

Remark 3.3. The corollary fails without the condition on a $A^{q,A}_n$-module $M$. In fact, for $1 \leq i \leq n$, there exists a $Z$-torsion finitely generated $A^{q,A}_n$-module $M$ with $\text{GKdim}_{A^{q,A}_n}(M) = i$. Put $A = A^{q,A}_n$. Let $L = A_{y_1+1} + \cdots + A_{y_n} + Ax_1 + \cdots + Ax_n$. The $A$-module $M = A/L$ has the filtration $B'(M)$ induced by the filtration $B(A)$ of $A$. Thus $B'(M) = (B_s(A) + L)/L$ is isomorphic as a vector space to

$$\bigoplus_{\alpha_1 + \cdots + \alpha_i \leq s} ky_1^{\alpha_1} \cdots y_i^{\alpha_i}.$$

Hence $\dim_k B'_s(M) = \binom{i+s}{s}$. This implies that $\text{GKdim}_A(M) = i$.

Another Bernstein’s inequality for quantized Weyl algebras has been considered by Demidov in [D].

References


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