Inverse and Direct Images for Quantum Weyl Algebras

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In [18] Wess and Zumino gave a method for constructing noncommutative differential calculus (or de Rham complex) on the quantum affine space associated to a Hecke symmetry \( R \). Also, they constructed the corresponding algebra of linear differential operators. Since the algebra of linear differential operators on the \( n \)-dimensional affine space is the \( n \)-th Weyl algebra, this algebra is regarded as a quantum analogue of the Weyl algebra, and called the quantum Weyl algebra (associated to \( R \)).

Let \( R_{q,P} \) be the multiparameter \( R \)-matrix of the quantum deformation of \( \text{GL}_n \) parameterized by a scalar \( q \) and an \( n \times n \) matrix \( P = (p_{ij}) \) in [3]. For the quantum Weyl algebra \( A_n(q,P) \) associated to \( R_{q,P} \), Demidov [6] and Rigal [15] consider quantum versions of classical theory of the Weyl algebras including Bernstein’s inequality. And, some ring-theoretic properties of \( A_n(q,P) \) have been studied in [1, 2, 9, 10, 11] etc. In [11] Jordan constructed a simple localization \( B_n(q,P) \) of \( A_n(q,P) \), which is a better analogue of the Weyl algebra \( A_n \) from the point of view of noncommutative ring theory.

The purpose of this paper is to define an analogue of the inverse and direct images for the quantum Weyl algebra \( A_n(q,P) \), and to investigate their properties. In particular, we prove a quantum analogue of Kashiwara’s theorem (Section 4), and consider preservation of holonomicity under inverse and direct images (Section 5).

Throughout this paper we fix a ground field \( K \) and let \( q \) be a nonzero element of \( K \) such that \( q^2 \) is not a root of unity, and we use the following \( q \)-integer notation:

\[
[i] = \frac{q^{2i} - 1}{q^2 - 1}, \quad [[i]] = \frac{q^{-2i} - 1}{q^{-2} - 1} \quad (i : \text{a nonnegative integer}).
\]

In this paper, we use the terminology and the results of [14] for noncommutative
Quantum Weyl Algebras


1 Preliminaries

Let \( V \) be an \( n \)-dimensional vector space. Assume that a non-degenerate linear transformation \( R : V \otimes V \to V \otimes V \) is a Hecke symmetry, that is, satisfies the Yang-Baxter equation

\[
R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},
\]

and the Hecke condition

\[
(R - q)(R + q^{-1}) = 0
\]

for some \( q \in K \setminus \{0\} \), where \( R_{12} = R \otimes \text{id}_V \), \( R_{23} = \text{id}_V \otimes R \). For fixed basis \( \{e_1, \cdots, e_n\} \) of \( V \), we write

\[
R(e_i \otimes e_j) = R_{kl}^{ij} e_k \otimes e_l \quad (R_{kl}^{ij} \in K).
\]

**EXAMPLE 1.1 ([10, Example 2.1; 12, Example VIII.1.3]).** Let \( P = (p_{ij}) \) be an \( n \times n \) matrix over \( K \setminus \{0\} \) such that \( p_{ii} = 1 \), \( p_{ji} = p_{ij}^{-1} \) for each \( i, j \). Define the multiparameter \( R \)-matrix \( R_{q,P} \) by

\[
(R_{q,P})_{ij}^{kl} = \delta_i^l \delta_j^k (p_{ij} + (q - p_{ij})\delta_{ij}) + (q - q^{-1})\delta_i^l \delta_j^k \theta(j, i),
\]

where

\[
\theta(i, j) = \begin{cases} 
1 & \text{if } i > j, \\
0 & \text{if } i \leq j.
\end{cases}
\]

The quantum affine space \( K_{R}[X] \) associated to a Hecke symmetry \( R \) is the \( K \)-algebra generated by \( x^1, \cdots, x^n \) with relations

\[
P_{kl}^{ij} x^k x^l = qx^i x^j.
\]

For the Hecke symmetry \( R_{q,P} \) in Example 1.1, we write \( K_{q,P}[X] \) for \( K_{R_{q,P}}[X] \). Then the relations of \( K_{q,P}[X] \) are as follows:

\[
x^i x^j = qp_{ij} x^j x^i \quad (i < j).
\]

In [18] Wess and Zumino construct examples of noncommutative differential calculus on the quantum affine space.
DEFINITION 1.2 ([18]). The Wess-Zumino differential calculus $\Omega(R)$ on $K_R[X]$ is the $K_R[X]$-ring with generators $\xi^1, \cdots, \xi^n$ subject to the relations

$$\xi^i \xi^j = -q R^{ij}_{kl} \xi^k \xi^l,$$

$$x^i \xi^j = q R^{ij}_{kl} x^k \xi^l.$$

Put $\Omega^0 = K_R[X]$, $\Omega^1 = \bigoplus_{i=1}^n K_R[X] \xi^i$ and $\Omega^l = (\Omega^1)^l$. Then $\Omega(R) = \bigoplus_{l \geq 0} \Omega^l$ is a differential graded algebra (for short, DG-algebra) with a differential map $d : \Omega(R) \to \Omega(R)$ such that

$$d(x^i) = \xi^i.$$

In [7, Prop.11.4] it is shown that $\Omega(R) \cong K_R[X] \otimes K_R[\xi]$ (as $K$-vector spaces), where $K_R[\xi]$ denotes the $K$-subalgebra of $\Omega(R)$ generated by $\xi^1, \cdots, \xi^n$. Hence, if $\{a_i\}_{i \in I}$ form $K$-basis of $K_R[\xi]$, then $\Omega(R)$ is a free left $K_R[X]$-module with $K_R[X]$-basis $\{1 \otimes a_i\}_{i \in I}$.

DEFINITION 1.3 ([18, 10]). The quantum Weyl algebra $A_n(R)$ associated to $R$ is defined as the $K_R[X]$-ring generated by $\partial_1, \cdots, \partial_n$ with the relations

$$R^{ik}_{ji} \partial_k \partial_l = q \partial_l \partial_j,$$

$$\partial_i x^j = \delta^j_i + q R^{ij}_{kl} x^k \partial_l.$$

In addition, commutation relations between $\partial_i$ and $\xi^j$ are given by

$$\partial_i \xi^j = q^{-1} (R^{-1})^{jk}_{il} \xi^k \partial_l.$$

EXAMPLE 1.4. Consider the Hecke symmetry $R_{q,P}$ in Example 1.1. We write $\Omega(q, P)$ and $A_n(q, P)$ for $\Omega(R_{q,P})$ and $A_n(R_{q,P})$, respectively. The relations of $\Omega(q, P)$ are

$$\begin{align*}
(\xi^i)^2 &= 0, \quad \xi^i \xi^j = -q^{-1} p_{ij} \xi^i \xi^j \quad (i < j), \\
x^i \xi^j &= q p_{ij} \xi^i x^j + (q^2 - 1) \xi^i x^j \quad (i < j), \\
x^i \xi^j &= q p_{ij} \xi^j x^i \quad (i > j), \\
x^i \xi^i &= q^2 \xi^i x^i.
\end{align*}$$

The relations of $A_n(q, P)$ are given by

$$\begin{align*}
\partial_i \partial_j &= q^{-1} p_{ij} \partial_j \partial_i \quad (i < j), \\
\partial_i x^j &= q p_{ij} x^j \partial_i \quad (i \neq j), \\
\partial_i x^i &= 1 + q^2 x^i \partial_i + (q^2 - 1) \sum_{j > i} x^j \partial_j.
\end{align*}$$
Put $F_0(A_n(q, P)) = K$, $F_1(A_n(q, P)) = K1 + Kx^1 + \cdots + Kx^n + K\partial_1 + \cdots + K\partial_n$ and $F_k(A_n(q, P)) = F_1(A_n(q, P))^k$ for $k \geq 1$. Then $A_n(q, P)$ is a filtered $K$-algebra with the filtration $F(A_n(q, P)) = \{F_k(A_n(q, P))\}_{k \geq 0}$.

When $P$ is the $n \times n$ matrix whose entries are all 1, we write $A_n^t$ for $A_n(q, P)$.

From the above relations one can obtain the following lemma.

**LEMMA 1.5.** Let $P = (p_{ij})$ be an $n \times n$ matrix as in Example 1.1. Denote by $P^t$ its transposed matrix.

1. There exists a DG-algebra isomorphism $\sigma : \Omega(q, P)^{op} \to \Omega(q^{-1}, P^t)$ such that $\sigma(x^i) = x^i$, $\sigma(\xi^i) = \xi^i$ (1 \leq i \leq n).

Here $\Omega(q, P)^{op}$ denotes the opposite DG-algebra of the DG-algebra $\Omega(q, P)$.

2. There exists a $K$-algebra anti-isomorphism $\tau : A_n(q, P) \to A_n(q^{-1}, P^t)$ such that $\tau(x^i) = x^i$, $\tau(\partial_i) = -q^{-2(n-i+1)}\partial_i$ (1 \leq i \leq n).

In [11] Jordan constructed a simple localization of $A_n(q, P)$. For 1 \leq i \leq n, let $z_i = \partial_ix^i - x^i\partial_i (= 1 + (q^2 - 1) \sum_{j \geq 1} x^j\partial_j)$. The subset $\mathcal{Z} = \{z_1^{a_1} \cdots z_n^{a_n}\}_{a_1, \ldots, a_n \geq 0}$ is an Ore set in $A_n(q, P)$ [11, 3.1]. We denote by $B_n(q, P)$ the localization of $A_n(q, P)$ at $\mathcal{Z}$. In [11, Thm. 3.2] it is proved that the localization $B_n(q, P)$ is simple of Krull and global dimension $n$ like the Weyl algebra $A_n$ in characteristic zero.

**REMARK 1.6.** The localization $B_n(q, P)$ has another generators. For 1 \leq i \leq n put $\hat{\partial}_i = z_{i+1}^{-1}\partial_i$, where $z_{n+1} = 1$. Then, as described in [2, 1.7], the $K$-algebra $B_n(q, P)$ is generated by $x^1, \ldots, x^n, \hat{\partial}_1, \ldots, \hat{\partial}_n, \hat{z}_1, \ldots, \hat{z}_n$, where $\hat{z}_i = \hat{\partial}_ix^i - x^i\hat{\partial}_i (= \hat{z}_{i+1}^{-1}\hat{z}_i)$. By [2, 1.5] the relations of these elements are as follows:

- $\hat{\partial}_i\hat{\partial}_j = qp_{ij}\hat{\partial}_j\hat{\partial}_i$, $\hat{\partial}_i x^j = q^{-1}p_{ij} x^i \hat{\partial}_j$ (i < j),
- $\hat{\partial}_i x^j = q p_{ij} x^i \hat{\partial}_j$ (i > j), $\hat{\partial}_i x^i = 1 + q^2 x^i \hat{\partial}_i$,
- $\hat{z}_i x^j = (\delta_i^j(q^2 - 1) + 1)x^i \hat{z}_j$, $\hat{z}_i \hat{\partial}_j = (\delta_{ij}(q^2 - 1) + 1) \hat{\partial}_j \hat{z}_i$,
- $\hat{\partial}_i \hat{\partial}_j = \hat{\partial}_j \hat{\partial}_i$.

Put $F_0(B_n(q, P)) = K$, $F_1(B_n(q, P)) = K1 + Kx^1 + \cdots + Kx^n + K\hat{\partial}_1 + \cdots + K\hat{\partial}_n + K\hat{z}_1^{-1} + \cdots + K\hat{z}_n^{-1}$ and $F_k(B_n(q, P)) = F_1(B_n(q, P))^k$ for $k \geq 1$. Then $B_n(q, P)$ is a filtered $K$-algebra with the filtration $F(B_n(q, P)) = \{F_k(B_n(q, P))\}_{k \geq 0}$. 
In [15] Rigal prove the following fact on the Gelfand-Kirillov dimension of modules over $B_n(q, P)$.

**Bernstein’s inequality [15, Thm.3(c)].** For a finitely generated nonzero left $B_n(q, P)$-module $M$, its Gelfand-Kirillov dimension $\text{GKdim}(M) \geq n$.

Following [15] we say that a finitely generated left $B_n(q, P)$-module $M$ is **holonomic** if $M = 0$ or $\text{GKdim}(M) = n$.

We say that an element $u$ of a left $A_n(q, P)$-module $M$ is **$\mathbb{Z}$-torsion** if there exists $w \in \mathbb{Z}$ such that $wu = 0$. We say that $M$ is **$\mathbb{Z}$-torsionfree** if $M$ has no nonzero $\mathbb{Z}$-torsion elements.

**Lemma 1.7.** Let $M$ be a finitely generated left $A_n(q, P)$-module. If $M$ is $\mathbb{Z}$-torsionfree, then

$$\text{GKdim}_{A_n(q, P)}(M) = \text{GKdim}_{B_n(q, P)}(B_n(q, P) \otimes_{A_n(q, P)} M).$$

*Proof.* See the proof of [8, Cor.3.2].

For a left $A_n(q, P)$-module $M$, let $T(M)$ be the submodule consisting of the $\mathbb{Z}$-torsion elements. Since $M/T(M)$ is $\mathbb{Z}$-torsionfree, $\text{GKdim}_{A_n(q, P)}(M/T(M)) \geq n$ or $M/T(M) = 0$ by the Lemma and the Bernstein’s inequality. We say that $M$ is **holonomic** if $\text{GKdim}_{A_n(q, P)}(M/T(M)) = n$ or $M/T(M) = 0$.

From the relations of $A_n(q, P)$ described in Example 1.4, one sees that

$$A_n(q, P)/\sum_{i=1}^{n} A_n(q, P)\partial_i \cong K_{q,P}[X] \quad \text{(as } K\text{-vector spaces}).$$

Via this linear isomorphism, $K_{q,P}[X]$ has a left $A_n(q, P)$-module structure. Then, $\partial_i$ acts on $K_{q,P}[X]$ as the $q$-difference operator:

$$\partial_i \cdot f(x^i) = \frac{f(q^2x^i) - f(x^i)}{q^2x^i - x^i} \quad (f(x^i) \in K[x^i]),$$

$$\partial_i \cdot f(x^j) = 0 \quad (f(x^j) \in K[x^j], \text{ where } j \neq i).$$

Each $z_i$ acts as a $K$-linear automorphism of $K_{q,P}[X]$ such that $(x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n} \mapsto q^{2(\alpha_1+\cdots+\alpha_n)}(x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$, so that $K_{q,P}[X]$ naturally becomes a left $B_n(q, P)$-module. Similarly, the $K$-subalgebra $K_{q,P}[\partial]$ of $A_n(q, P)$ generated by $\partial_1, \cdots, \partial_n$ has a left $B_n(q, P)$-module structure via the linear isomorphism

$$A_n(q, P)/\sum_{i=1}^{n} A_n(q, P)x^i \cong K_{q,P}[\partial].$$
EXAMPLE 1.8. Both the left $A_n(q, P)$-modules $K_{q,P}[X]$ and $K_{q,P}[\partial]$ are holonomic. In fact, $K_{q,P}[X]$ has the good filtration $\mathcal{F} = \{\mathcal{F}_k\}_{s \geq 0}$ such that $\mathcal{F}_k = \mathcal{F}_k(A_n(q, P)) \cdot 1$ for $k \geq 0$. Obviously, $\mathcal{F}_k$ is isomorphic as a vector space to

$$\bigoplus_{\alpha_1 + \cdots + \alpha_n \leq k} K(x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$$

whose dimension equals $\binom{n+k}{n}$. This implies that $\text{GKdim}(K_{q,P}[X]) = n$, which means that $K_{q,P}[X]$ is holonomic since $K_{q,P}[X]$ is $\mathbb{Z}$-torsionfree. Similarly, the holonomicity of $K_{q,P}[\partial]$ can be shown.

2 Quantum Matrix Group Action and Coaction on Quantum Weyl Algebras

DEFINITION 2.1. Let $R$ be a Hecke symmetry. $M(R)$ is the $K$-algebra with $n^2$ generators $t^i_j$ $(1 \leq i, j \leq n)$ subject to the relations

$$R_{\alpha \beta}^{ij} t^\alpha_k t^\beta_l = R_{\alpha \beta}^{kl} t^\alpha_i t^\beta_j.$$

$M(R)$ has a bialgebra structure with the comultiplication $\Delta$ and the counit $\varepsilon$ such that

$$\Delta(t^i_j) = t^i_\alpha \otimes t^\alpha_j, \quad \varepsilon(t^i_j) = \delta^i_j.$$

Denote by $H(R)$ the Hopf envelope of $M$. Thus there exists a bialgebra morphism $\Psi : M(R) \to H(R)$ such that, for any bialgebra morphism $\psi : M(R) \to H$ with $H$ being a Hopf algebra, there exists a Hopf algebra morphism $\tilde{\psi} : H(R) \to H$ with $\psi = \tilde{\psi} \circ \Psi$. Such a Hopf algebra $H(R)$ always exists. See [13, Ch.7] for details.

EXAMPLE 2.2. Let $R_{q,P}$ be the Hecke symmetry in Example 1.1. Consider the bialgebra $M(R_{q,P})$. The commutation relations between the generators $t^i_j$ are as follows ([3]) :

$$p_{ik} t^k_i t^l_i = q t^l_i t^k_i, \quad p_{ji} t^k_j t^k_i = q t^k_j t^k_i,$$

$$p_{ji} t^l_j t^k_i = p_{ik} t^k_i t^l_j, \quad p_{ji} t^l_j t^k_i - p_{kl} t^l_i t^k_j = (q - q^{-1}) p_{ji} p_{kl} t^k_i t^l_j,$$

where $i < j$ and $l < k$. This bialgebra is regarded as a $q$-deformation of the coordinate ring $O(M_n)$ of the space of $n \times n$ matrices, and usually denoted by $O_q(M_n)$. The Hopf envelope $H(R_{q,P})$ of $M(R_{q,P})$ is obtained as $M(R_{q,P})[\text{det}_q^{-1}]$, where $\text{det}_q$ is the quantum determinant defined in [3].
The bialgebra $M(R)$ has a cobraided structure $\langle \ , \ \rangle : M(R) \times M(R) \rightarrow K$ such that

$$\langle t^i_j, t^k_l \rangle = q(R^{-1})_{ji}^{kl}.$$

Thus $\langle \ , \ \rangle$ is a bilinear, and satisfies that

$$\langle a, bc \rangle = \langle a^{(1)}, c \rangle \langle a^{(2)}, b \rangle,$$

$$\langle ab, c \rangle = \langle a, c^{(1)} \rangle \langle b, c^{(2)} \rangle,$$

$$b^{(1)}a^{(1)} \langle a^{(2)}, b^{(2)} \rangle = \langle a^{(1)}, b^{(1)} \rangle a^{(2)} b^{(2)}$$

for all $a, b, c \in M(R)$, where we use the Sweedler notation $\Delta(a) = a^{(1)} \otimes a^{(2)}$ etc. See [12, Thm. VIII.6.4].

Throughout this paper we assume that the cobraided structure is extended to $H(R)$. This assumption holds for $R_{q,p}$.

**Lemma 2.3.** (1) There exists a right $H(R)^{op}$-comodule algebra structure $\rho$ on $A_n(R)$ such that

$$\rho(x^i) = x^\alpha \otimes S(t^i_\alpha),$$

$$\rho(\partial_i) = \partial_\alpha \otimes t^\alpha_i,$$

for $1 \leq i \leq n$, where $S$ denotes the antipode of the Hopf algebra $H(R)$.

(2) There exists a left $H(R)$-module algebra structure on $A_n(R)$ such that

$$h \cdot D = \langle h, D^{(1)} \rangle D^{(0)}$$

for $h \in H(R)$, $D \in A_n(R)$, where $\rho(D) = D^{(0)} \otimes D^{(1)}$.

**Proof.** This lemma can be directly verified. Also, see [16].

Define the $K$-algebra automorphism $\varphi : K_R[X] \rightarrow K_R[X]$ by

$$\varphi(x^i) = q^2 x^i \quad (1 \leq i \leq n).$$

For later use, we need the following lemma.

**Lemma 2.4.** For the Wess-Zumino calculus $\Omega(R)$, it holds that

$$f \xi^i = \langle f^{(1)}, t^i_\alpha \rangle \xi^\alpha \varphi(f^{(0)})$$

for all $f \in K_R[X]$. 

Proof. We may assume that $f$ is a monomial. Clearly, it is true for $f = 1$. If $f = x^j g$ with $g \in K_R[X]$, then

$$x^j g \xi^i = \langle g(1), t^{i}_n \rangle x^j \xi^\alpha \varphi(g(0))$$

$$= \langle g(1), t^{i}_n \rangle \langle x^j(1), t^{0}_k \rangle \xi^k \varphi(x^j(0)) \varphi(g(0))$$

$$= \langle f(1), t^{i}_n \rangle \xi^\alpha \varphi(f(0)),$$

since $R_{kl}^{ij} = q \langle S(t^j_i), t^{0}_k \rangle$. Therefore, the lemma follows. \qed

For $k \geq 0$, following [16, 17], we define the twisted bracket $[\ , \ ]_k : A_n(R) \times K_R[X] \to A_n(R)$ by

$$[D, f]_k = Df - \varphi^k(f(0))(f(1) \cdot D) \quad (D \in A_n(R), f \in K_R[X]).$$

One can verify that, $[\partial_i, f]_1 \in K_R[X]$ for all $f \in K_R[X]$. Using this twisted bracket, the left $A_n(R)$-action on $K_R[X]$ is described as follows:

$$x^i \cdot f = x^i f, \quad \partial_i \cdot f = [\partial_i, f]_1 \quad (f \in K_R[X]).$$

Futher, the following twisted Leibniz rule holds:

$$\partial_i(fg) = \partial_i(f)g + \varphi(f(0))(f(1) \cdot \partial_i)(g) \quad (f, g \in K_R[X]).$$

We end this section by collecting results needed later.

**Lemma 2.5.** In the Wess-Zumino calculus $\Omega(R)$, it holds that

$$d(f) = \xi^\alpha \partial_\alpha(f)$$

for all $f \in K_R[X]$.

**Lemma 2.6 ([16]).** In $K_R[X] \otimes H(R)$, it holds that

$$D(f)(0) \otimes D(f)(1) = D(0)(f(0)) \otimes D(1)f(1)$$

for all $D \in A_n(R), f \in K_R[X]$. 
3 Inverse and Direct Images of Modules over Quantum Weyl Algebras

The main purpose of this section is to define a quantum analogue of the inverse and direct images for quantum Weyl algebras. We refer to [4, 5] for the inverse and direct images for the classical Weyl algebra.

Fix another nonnegative integer \(m\). Let \(V'\) be an \(m\)-dimensional vector space, and \(R' : V' \otimes V' \to V' \otimes V'\) a Hecke symmetry for \(q\). For the algebras \(\Omega(R')\) and \(A_m(R')\) the variables, differentials and derivatives are denoted by \(y^1, \ldots, y^m, \eta^1, \ldots, \eta^m\) and \(\partial'_1, \ldots, \partial'_m\), respectively. The generators of \(H(R')\) is denoted by \(t^i_j (1 \leq i, j \leq m)\), and we denote by \(\langle \langle \cdot, \cdot \rangle \rangle\) the cobraided structure on \(H(R')\) described in Section 2.

Let \(F : \Omega(R') \to \Omega(R)\) be a DG-algebra morphism. Thus, in particular, the restriction of \(F\) to \(K_{R'}[Y]\) is a \(K\)-algebra morphism from \(K_{R'}[Y]\) to \(K_R[X]\). Then \(K_R[X]\) has a right \(K_{R'}[Y]\)-module structure via \(F\).

Let \(M\) be a left \(A_m(R')\)-module (so \(M\) is also a left \(K_{R'}[Y]\)-module).

**DEFINITION 3.1.** The **inverse image** of \(M\) under \(F\) is the left \(K_R[X]\)-module

\[
F^* M = K_R[X] \otimes_{K_{R'}[Y]} M.
\]

**LEMMA 3.2.** For \(1 \leq i \leq n\), there exists a \(K\)-linear morphism \(\phi_i : F^* M \to F^* M\) such that, for \(f \in K_R[X], u \in M\),

\[
\phi_i(f \otimes u) = \partial_i(f) \otimes u + \varphi(f(0))(f(1) \cdot \partial_i)(F^i) \otimes \partial'_i u,
\]
where \(F^i = F(y^i)\).

**Proof.** For \(1 \leq i \leq n\), we define the \(K\)-linear morphism \(\Phi_i : K_R[X] \otimes M \to F^* M\) by

\[
\Phi_i(f \otimes u) = \partial_i(f) \otimes u + \varphi(f(0))(f(1) \cdot \partial_i)(F^i) \otimes \partial'_i u
\]
for \(f \in K_R[X], u \in M\). If it holds that, for \(f \in K_R[X], g \in K_{R'}[Y], u \in M\),

\[
\Phi_i(f F(g) \otimes u) = \Phi_i(f \otimes gu), \tag{3.3}
\]
then \(\Phi_i\) induces the desired linear morphism \(\phi_i\). Therefore it suffices to show that the equality (3.3) holds.

By a direct computation one sees that the left-hand side of (3.3) equals

\[
\partial_i(f) F(g) \otimes u + \varphi(f(0))(f(1) \cdot \partial_i)(F(g)) \otimes u
+ \varphi(f(0)f(g)(0))(f(1) \cdot \partial_i)(F^i)) \otimes \partial'_i u,
\]
and the right-hand side of (3.3) equals
\[\partial_i (f) \otimes gu + \varphi(f_{(0)})(f_{(1)} \cdot \partial_i)(F^i) \otimes \partial'_i (g) u \]
\[+ \varphi(f_{(0)})(f_{(1)} \cdot \partial_i)(F^i) \otimes \varphi'(g_{(0)})(g_{(1)} \cdot \partial'_i) u,\]
where \(\varphi'\) is the \(K\)-algebra automorphism of \(K_R[Y]\) such that \(\varphi'(y^i) = q^2 y^i\). Since \(F(d(g)) = d(F(g))\), it follows from Lemma 2.5 and the fact that \(\xi^1, \ldots, \xi^n\) are \(K_R[X]\)-linearly independent, that
\[(h \cdot \partial_i)(F^i) F(\partial'_i (g)) = (h \cdot \partial_i)(F(g)) \quad (h \in H(R)).\]
Hence it is sufficient to show that, for each \(\beta\),
\[\varphi(f_{(0)} F(g_{(0)}))(F(g_{(1)}) f_{(1)} \cdot \partial_i)(F^i) \varphi'(g_{(0)})) = \langle \langle g_{(1)}, \tilde{t}^\beta_i \rangle \rangle \varphi(f_{(0)})(f_{(1)} \cdot \partial_i)(F^i) F(\varphi'(g_{(0)})) \quad (3.4)\]
From Lemma 2.4 for \(\Omega(R')\) it follows that
\[g \eta^\beta = \langle \langle g_{(1)}, \tilde{t}^\beta_i \rangle \rangle \eta^i \varphi'(g_{(0)}).\]
Applying \(F\) to this equation, one obtain
\[F(g) \xi^\alpha \partial_{\alpha}(F^\beta) = \langle \langle g_{(1)}, \tilde{t}^\beta_i \rangle \rangle \xi^i \partial_i(F^i) F(\varphi'(g_{(0)})).\]
Again, by Lemma 2.4 it follows that, for each \(l, \beta\),
\[\xi^\alpha \varphi(F(g_{(0)}))(F(g_{(1)} \cdot \partial_{\alpha})(F^\beta) = \xi^\alpha \langle \langle g_{(1)}, \tilde{t}^\beta_i \rangle \rangle \partial_{\alpha}(F^i) F(\varphi'(g_{(0)})).\]
Since \(\{\xi^1, \ldots, \xi^n\}\) is \(K_R[X]\)-linearly independent, the equation (3.4) holds. \(\square\)

**PROPOSITION 3.5.** Let notations be as above. If it holds that, for \(1 \leq i, j \leq n, u \in M\),
\[\varphi^2(F_{(0)}^i)(F_{(1)}^j \cdot (\partial_i \partial_j))(F^k) \otimes \partial'_k u = q(R')_{\psi_{ij}}^l \partial_i \partial_j(F^\psi) F^\mu \otimes \partial'_l u \quad \text{(in } F^* M\text{), (}\ast\text{)}\]
then the inverse image \(F^* M\) of \(M\) under \(F\) is a left \(A_n(R)\)-module with the action defined by
\[x^i \cdot (f \otimes u) = x^i f \otimes u,\]
\[\partial_i \cdot (f \otimes u) = \phi_i(f \otimes u) = \partial_i(f) \otimes u + \varphi(f_{(0)})(f_{(1)} \cdot \partial_i)(F^i) \otimes \partial'_i u\]
for \(f \in K_R[X], u \in M\).
Proof. We have to prove the following equalities:

\[
\phi_i x^j (f \otimes u) = (\delta^j_i + q R^{ij\alpha}_{\alpha} x^\alpha \phi_\beta)(f \otimes u), \tag{3.6}
\]

\[
R^{3\alpha}_{ji} \phi_\alpha \phi_\beta (f \otimes u) = q \phi_i \phi_j (f \otimes u), \tag{3.7}
\]

for all \( f \in K_R[X], \ u \in M, \ 1 \leq i, j \leq n. \)

It is direct to show that the equality (3.6) holds. In fact, by a direct computation one obtain

\[
\phi_i x^j (f \otimes u) = \partial_i (x^j) f \otimes u + \varphi(x^\alpha)((S(t^i_\alpha) \cdot \partial_i)(f)) \otimes u + \varphi(x^\alpha f(0))(f(t^i_\alpha) \cdot \partial_i)(F^j) \otimes \partial_i u
\]

\[
= \delta^j_i f \otimes u + q^2 S(t^i_\alpha) x^\alpha \partial_\beta(f) \otimes u + q^2 S(t^i_\alpha) x^\alpha \varphi(f(0))(f(1) \cdot \partial_\beta)(F^j) \otimes \partial_i u,
\]

which equals \((\phi_i + a R^{ij\alpha}_{\alpha} x^\alpha \phi_\beta)(f \otimes u).\)

We will prove that the equation (3.7) holds. We may assume that \( f \) is a monomial of \( K_R[X]. \) By a direct computation, one obtain, for each \( \alpha, \beta, \)

\[
\phi_\alpha \phi_\beta (f \otimes u) = (\partial_\alpha \partial_\beta)(f) \otimes u + \varphi(\partial_\beta(f(0)))(\partial_\alpha(f(1)) \cdot \partial_\alpha)(F^j) \otimes \partial_i u
\]

\[
+ \partial_\alpha(\varphi(f(0)))(f(1) \cdot \partial_\beta)(F^j) \otimes \partial_i u
\]

\[
+ \varphi(\varphi(f(0))(f(1) \cdot \partial_\beta)(F^j)(0))(\varphi(f(0))(f(1) \cdot \partial_\beta)(F^j)(1) \cdot \partial_\alpha)(F^{k}) \otimes \partial_i \partial_k u.
\]

Therefore it suffices to show that the following equalities holds:

\[
R^{3\alpha}_{ji} \varphi(\partial_\beta(f(0))(\partial_\alpha(f(1)) \cdot \partial_\alpha)(F^j) + R^{3\alpha}_{ji} \partial_\alpha \varphi(f(0))(f(1) \cdot \partial_\beta)(F^{j})
\]

\[
= q \varphi(\partial_j (f(0))(\partial_j(f(1)) \cdot \partial_j)(F^j) + q \partial_j \varphi(f(0))(f(1) \cdot \partial_j)(F^j), \tag{3.8}
\]

\[
R^{3\alpha}_{ji} \varphi(\varphi \cdot f(0))(\varphi(f(0))(f(1) \cdot \partial_\alpha)(F^j)
\]

\[
= q \varphi(\varphi(f(0))(f(1) \cdot \partial_\alpha)(F^j), \tag{3.9}
\]

\[
R^{3\alpha}_{ji} \varphi((\varphi(f(0))(f(1) \cdot \partial_\beta)(F^j))(0))(\varphi(f(0))(f(1) \cdot \partial_\beta)(F^j))(1) \cdot \partial_\alpha)(F^{k}) \otimes \partial_i \partial_k u
\]

\[
= (R^{k\nu}_{\psi\mu} \varphi((\varphi(f(0))(f(1) \cdot \partial_j)(F^{\psi}))(0))(\varphi(f(0))(f(1) \cdot \partial_j)(F^{\psi}))(1) \cdot \partial_i)(F^\mu) \tag{3.10}
\]

\[
\otimes \partial_j \partial_k u.
\]
First, we shall prove the equation (3.8). By Lemma 2.6, one obtain
\[
\varphi(\partial_\beta(f)_0)(\partial_\beta(f)_1 \cdot \partial_\alpha)(F^l)
= q^{-1}(R^{-1})_{\alpha\gamma}^\beta \varphi(f_0))((f_1 \cdot \partial_\alpha)(F^l),
\]
since \(\varphi \partial_\gamma(g) = q^{-2} \partial_\gamma \varphi(g)\) for all \(g \in K R[X]\). Noting that \(R = ((q - q^{-1})I + R^{-1})\), one sees that the left-hand side of (3.8) equals
\[
R_{ji}^{\theta\alpha} q^{-1}(R^{-1})^{\alpha\gamma} \varphi(f_0))((f_1 \cdot \partial_\alpha)(F^l)
+ ((q - q^{-1})I + R^{-1})^{\beta\alpha} \varphi(f_0))((f_1 \cdot \partial_\beta)(F^l),
\]
which is equal to the right-hand side of (3.8). Hence (3.8) holds.

Next we consider the equation (3.9). Clearly, the equation (3.9) holds for \(f = 1\). We assume that \(f = x^g\) with \(g\) a monomial. Then the left-hand side of (3.9) is equal to
\[
R_{ji}^{\theta\alpha} \varphi(x^g)_0(\varphi(x^g)_1 \cdot \partial_\alpha)((x^g)_1 \cdot \partial_\beta)(F^l)
= q^2(S_t^\alpha R_{ji}^{\beta\alpha} t^\alpha t^\alpha)(\alpha \cdot \partial_\alpha)(\alpha \cdot \partial_\beta)(F^l)
= q^2(S_t^\alpha R_{ji}^{\beta\alpha} t^\alpha t^\alpha)(\alpha \cdot \partial_\alpha)(\alpha \cdot \partial_\beta)(F^l),
\]
which equals
\[
q^2(\alpha \cdot \partial_\alpha)(\alpha \cdot \partial_\beta)(F^l),
\]
by the induction hypothesis for \(g\). This equals the right-hand side of (3.9).

It remains to show that (3.10) holds. We note that \(\varphi(f)_0 \otimes \varphi(f)_1 = \varphi(f_0) \otimes f_1\). One sees that, for each \(\alpha, \beta, l, k\),
\[
\varphi((\varphi(f_0)_0, (f_1 \cdot \partial_\beta)(F^l)_0)((\varphi(f_0)_0, (f_1 \cdot \partial_\beta)(F^l)_1 \cdot \partial_\alpha)(F^k)
= q^{-1}(R^{-1})_{\gamma\alpha}^\beta \varphi(f_0)_0 \partial_\alpha(\varphi(F^l_0)_0)(F^l_1 \cdot \partial_\alpha)(F^k).
\]

By (3.4) it follows that, for each \(l, k, \chi, \mu\),
\[
\varphi(F^l_0)(F^l_1 \cdot \partial_\chi)(F^k) = q(R')_{\chi\mu}^{\mu} \partial_\chi(F^\psi)F^\mu,
\]
which implies that
\[
\partial_\theta(\varphi(F^l_0))(F^l_1 \cdot \partial_\chi)(F^k)
= -\varphi^2(F^l_0)((F^l_1 \cdot (\partial_\theta \partial_\chi))(F^k)
+ q (R')_{\chi\mu}^{\mu} \partial_\theta \partial_\chi(F^\psi)F^\mu
+ (R')_{\chi\mu}^{\mu} (R^{-1})_{\chi\mu}^{\mu} \partial_\omega(\varphi(F^\psi_0)(F^l_1 \cdot \partial_\alpha)(F^\mu).
\]
Combining this equation with the equation (3.11) one sees that the left-hand side of (3.10) equals the right-hand side of (3.10) plus

\[ -q^{-1} \varphi(f(0)) \varphi^2(F_l(0))(F_{(1)}^l f_{(1)} \cdot (\partial_i \partial_j)) (F^k) \otimes \partial'_k u \]

\[ + (R'_l)^{lk}_{\psi \mu} \varphi(f(0))(f_{(1)} \cdot (\partial_i \partial_j)) (F^\psi) F^u \otimes \partial'_l \partial'_k u, \]

which is equals to zero by the assumption. Therefore (3.10) holds.

**COROLLARY 3.12.** If \( R = R_{q,P} \) and if \( p_{ij}^2 \neq q^2 \) for each \( i < j \), then the inverse image \( F^* M \) of \( M \) under \( F \) is a left \( A_n(R) \)-module with the action defined in Proposition 3.5.

**Proof.** We assume that \( p_{ij}^2 \neq q^2 \) for \( i < j \). It suffices to show that the assumption (*) in Proposition 3.5 holds.

In the proof of Proposition 3.5, it is known that

\[ R_{ji}^{ij} \phi_{\alpha} \phi_{\beta}(f \otimes u) \]

\[ = q \phi_{\alpha} \phi_{\beta}(f \otimes u) \]

\[ - q^{-1} \varphi(f(0)) \varphi^2(F_l(0))(F_{(1)}^l f_{(1)} \cdot (\partial_i \partial_j)) (F^k) \otimes \partial'_k u \]

\[ + (R'_l)^{lk}_{\psi \mu} \varphi(f(0))(f_{(1)} \cdot (\partial_i \partial_j)) (F^\psi) F^u \otimes \partial'_l \partial'_k u, \]

for \( f \in K_{q,P}[X] \), \( u \in M \), \( 1 \leq i, j \leq n \). On the other hand, when \( i = j \), by the definition of \( R_{q,P} \), the left-hand side of the above equation equals \( q \phi_{\alpha} \phi_{\beta}(f \otimes u) \). Therefore, (*) holds for \( i = j \).

We can assume that \( i < j \). Since \( d \) is a differential of \( \Omega(q,R) \), it follows that

\[ d^2 = -\xi^i \xi^j \partial_i \partial_j. \]

One sees that \( q(R_{q,P}^{i \mu} \xi^i \partial_i \partial_j)(F^\psi) F^u = 0 \). On the other hand, it follows from Lemma 2.4 that \( \xi^i \xi^j \varphi^2(F_l(0))(F_{(1)}^l f_{(1)} \cdot (\partial_i \partial_j)) (F^k) \otimes \partial'_k u, \]

From the relations of \( \Omega(q,R) \) and \( A_n(q,R) \), one obtains

\[ \sum_{i<j} (1 - q^2 p_{ji}^2) \xi^i \xi^j \varphi^2(F_l(0))(F_{(1)}^l f_{(1)} \cdot (\partial_i \partial_j)) (F^k) \]

\[ = \sum_{i<j} (1 - q^2 p_{ji}^2) q(R_{q,P}^{i \mu} \xi^i \partial_i \partial_j)(F^\psi) F^u, \]

Since \( \{ \xi^i \xi^j \mid i < j \} \) is \( K_{q,P}[X] \)-linearly independent, it follows that (*) holds for all \( i, j \).

In the rest of this paper, we will consider only matrices \( P \) satisfying the assumption in Corollary 3.12.
The inverse image $F^* A_m(q, P')$ naturally becomes a $A_n(q, P) \cdot A_m(q, P')$ bimodule. Following classical notation, we denote this bimodule by $D_{X \rightarrow Y}$. Then it follows that

$$F^* M \cong D_{X \rightarrow Y} \otimes_{A_m(q, P') M}$$

for any left $A_m(q, P')$-module $M$.

Given a DG-algebra morphism $F : \Omega(q, P') \rightarrow \Omega(q, P)$, by Lemma 1.5(1) we obtain a DG-algebra morphism $\Omega(q^{-1}, (P')^t) \rightarrow \Omega(q^{-1}, P^t)$ such that $y^j \mapsto F^j$ for $1 \leq j \leq m$. We also denote this morphism by $F$. Then by the above way we obtain the $(A_n(q^{-1}, P^t)\cdot A_m(q^{-1}, (P')^t))$ bimodule $D_{X \rightarrow Y} (= F^* A_m(q^{-1}, (P')^t))$. Define $D_{Y \rightarrow X}$ to be the $A_m(q, P')\cdot A_n(q, P)$ bimodule such that $D_{Y \rightarrow X} = D_{X \rightarrow Y}$ as a $K$-vector space, and that $A_m(q, P')\cdot A_n(q, P)$ bimodule action is defined by

$$D^j \star v \star D = \tau(D) \cdot v \cdot \tau(D')$$

for $D \in A_n(q, P), D' \in A_m(q, P')$ and $v \in D_{Y \rightarrow X}$, where $\cdot$ denotes the $A_n(q^{-1}, P^t)\cdot A_m(q^{-1}, (P')^t)$ bimodule action on $D_{X \rightarrow Y}$.

**DEFINITION 3.13.** Let $M$ be a left $A_n(q, P)$-module. The direct image $F_* M$ of $M$ under $F$ is the left $A_m(q, P')$-module $D_{Y \rightarrow X} \otimes_{A_n(q, P)} M$.

**EXAMPLE 3.14.** Fix nonnegative integers $n$ and $m$. Let $P = (p_{ij})_{1 \leq i, j \leq n+m}$ be an $(n+m) \times (n+m)$ matrix as in Example 1.1. For $\Omega(q, P)$ and $A_{n+m}(q, P)$, the variables, the differentials and the derivatives are denoted by

$$x^1, \ldots, x^n, y^1, \ldots, y^m, \xi^1, \ldots, \xi^n, \eta^1, \ldots, \eta^m \quad \text{and} \quad \partial_1, \ldots, \partial_n, \partial'_1, \ldots, \partial'_m$$

instead of $x^1, \ldots, x^{n+m}, \xi^1, \ldots, \xi^{n+m}$ and $\partial_1, \ldots, \partial_{n+m}$, where $y^j$ (resp. $\eta^j$, $\partial'_j$) plays the role of $x^{n+j}$ (resp. $\xi^{n+j}$, $\partial_{n+j}$) for $1 \leq j \leq m$.

For simplicity, we sometimes abbreviate the notation of the parameters if confusion does not occur. For example, we write $A_{n+m}$ for $A_{n+m}(q, P)$, $B_{n+m}$ for $B_{n+m}(q, P)$. Furthermore, we denote by $K[X,Y]$ the subalgebra of $A_{n+m}$ generated by $x^1, \ldots, x^n, y^1, \ldots, y^m$.

(1) Denote by $P'$ the $m \times m$ matrix with $(i, j)$-entry $p_{n+i,n+j}$. The variables, the differentials and the derivatives of $\Omega(q, P')$ and $A_m = A_m(q, P')$ are denoted by $y^1, \ldots, y^m, \eta^1, \ldots, \eta^m$ and $\partial'_1, \ldots, \partial'_m$, respectively.

Define the DG-algebra morphism $\pi : \Omega(q, P') \rightarrow \Omega(q, P)$ by

$$\pi(y^j) = y^j, \quad \pi(\eta^j) = \eta^j \quad (1 \leq j \leq m).$$
Let $M$ be a left $A_m$-module. From the definition, it is clear that

$$\pi^*M \cong K[X] \otimes M \quad \text{(as $K$-vector spaces)}$$

via the morphism defined by $fg \otimes u \mapsto f \otimes gu$ ($f \in K[X]$, $g \in K[Y]$, $u \in M$). Let $C$ be the $K$-subalgebra of $A_{n+m}$ generated by $x^1, \ldots, x^n, \partial_1, \ldots, \partial_n$. Note that $K[X]$ has a left $C$-module structure induced from the $A_{n+m}$-module structure of $K[X,Y]$ when we regard $K[X]$ as a $K$-subspace of $K[X,Y]$. By the above identification, the $A_{n+m}$-action on $K[X] \otimes A_m$ is as follows:

$$x^i(f \otimes u) = x^i f \otimes u, \quad y^j(f \otimes u) = \psi_j(f) \otimes y^j u,$$

$$\partial_i(f \otimes u) = \partial_i f \otimes u, \quad \partial'_j(f \otimes u) = \psi'_j(f) \otimes \partial'_j u$$

for $f \in K[X]$, $u \in M$, $1 \leq i \leq n$, $1 \leq j \leq m$, where $\psi_j$ and $\psi'_j$ are the $K$-algebra automorphism of $K[X]$ defined by

$$\psi_j(x^i) = q^{-1} p_{n+j,i} x^i, \quad \psi'_j(x^i) = q p_{i,n+j} x^i \quad (1 \leq i \leq n).$$

Furthermore it easily follows that

$$D_{X \times Y \rightarrow Y} \cong A_{n+m} / \sum_{i=1}^n A_{n+m} \partial_i \quad \text{(as $A_{n+m}$-$A_m$ bimodules)}.$$

Note that $z_i(f \otimes u) = z_i f \otimes u$, $z'_j(f \otimes u) = f \otimes z'_j u$ ($f \in K[X]$, $u \in M$, $1 \leq i \leq n$, $1 \leq j \leq m$), where $z'_j = \partial'_j y^j - y^j \partial' j$ for $1 \leq j \leq m$. Therefore, if $M$ is a left $A_{n+m}$-module, then $\pi^*M$ is naturally a left $B_{n+m}$-module.

(2) Let $\pi$ be the DG-algebra morphism defined in (1). From (1)

$$D_{Y \times X \rightarrow Y} = \pi_* A_{n+m} \cong A_{n+m} / \sum_{i=1}^n \partial_i A_{n+m} \quad \text{(as left $A_m$-$A_{n+m}$ bimodules)},$$

so, for any left $A_{n+m}$-module $M$,

$$\pi_* M \cong M / \sum_{i=1}^n \partial_i M \quad \text{(as left $A_m$-modules)}.$$

From the above isomorphism, it follows that, if $M$ is a left $B_{n+m}$-module, then $\pi_* M$ is naturally a left $B_m$-module.

(3) Let $P''$ be the $n \times n$ matrix with $(i,j)$-entry $p_{i,j}$. For $\Omega(q,P'')$ and $A_n = A_n(q,P'')$, we denote the corresponding variables, differentials and derivatives by $x^1, \ldots, x^n$, $\xi^1, \ldots, \xi^m$ and $\partial_1, \ldots, \partial_n$, respectively.
Define the DG-algebra morphism $\iota : \Omega(q, P) \to \Omega(q, P')$ by

\[
\begin{align*}
\iota(x^i) &= x^i, \quad \pi(\xi^i) = \xi^i \quad (1 \leq i \leq n), \\
\iota(y^j) &= 0, \quad \pi(\eta^j) = 0 \quad (1 \leq j \leq m).
\end{align*}
\]

For a left $A_{n+m}$-module $M$, one easily sees that $\iota^* M \cong M / \sum_{j=1}^m y_j M$ (as left $A_n$-modules)

via $K[X] \otimes_K [X,Y] M \to A_{n+m}/ \sum_{j=1}^m y_j M$, $f \otimes u \mapsto f u + \sum y^j M \ (f \in K[X], \ D \in M)$.

In particular, the $A_n$-$A_{n+m}$ bimodule $D_{X \to X \times Y} = \iota^* A_{n+m}$ is isomorphic to

$$A_{n+m} / \sum_{j=1}^m y_j A_{n+m}.$$

Note that, if $M$ is a left $B_{n+m}$-module, then $\iota^* M$ naturally becomes a left $B_n$-module.

(4) Let $\iota$ be the DG-algebra morphism defined in (3). It follows from (3) that

$$\iota_* M \cong (A_{n+m} / \sum_{j=1}^m A_{n+m} y_j^i) \otimes_{A_n} M \quad \text{(as left $A_{n+m}$-modules)}$$

for any $A_n$-module $M$. Similar to (1), $\iota_* M$ is isomorphic to the left $A_{n+m}$-module $K[\partial]^\perp \otimes M$ with the action as follows:

\[
\begin{align*}
x^i(f \otimes u) &= \psi_i(f) \otimes x^i u, \quad y^j(f \otimes u) = y^j f \otimes u, \\
\partial_i(f \otimes u) &= \psi_i(f) \otimes \partial_i u, \quad \partial'_j(f \otimes u) = \partial'_j f \otimes u, \\
z_i(f \otimes u) &= \psi'_i(f) \otimes z_i u, \quad z'_j(f \otimes u) = z'_j f \otimes u
\end{align*}
\]

for $f \in K[\partial]$, $u \in M$, $1 \leq i \leq n$, $1 \leq j \leq m$, where $\psi_j$, $\psi'_j$ are the $K$-algebra automorphism of $K[\partial]$ defined by

$$\psi_i(\partial'_j) = q^{-1} p_{i,n+j} \partial'_j, \quad \psi'_i(\partial'_j) = q^{-2} \partial'_j \quad (1 \leq j \leq m).$$

From the above observation, if $M$ is a left $B_n$-module, then $\iota_* M$ is naturally a left $B_{n+m}$-module.
LEMMA 3.15. Let notation be as in example 3.14.

(1) If $M$ is a finitely generated left $A_m(q, P')$-module, then $\pi^*M$ is a finitely generated $A_{n+m}(q, P)$-module, and

$$\text{GKdim}_{A_{n+m}(q, P)}(\pi^*M) = \text{GKdim}_{A_m(q, P')}(M) + n.$$ 

(2) If $M$ is a finitely generated left $B_m(q, P')$-module, then $\pi^*M$ is a finitely generated $B_{n+m}(q, P)$-module, and

$$\text{GKdim}_{B_{n+m}(q, P)}(\pi^*M) = \text{GKdim}_{B_m(q, P')}(M) + n.$$ 

In particular, $M$ is holonomic if and only if $\pi^*M$ is holonomic.

(3) If $M$ is a finitely generated left $A_n(q, P'')$-module, then $\iota_*M$ is a finitely generated $A_{n+m}(q, P)$-module, and

$$\text{GKdim}_{A_{n+m}(q, P)}(\iota_*M) = \text{GKdim}_{A_n(q, P'')}(M) + m.$$ 

(4) If $M$ is a finitely generated left $B_n(q, P'')$-module, then $\iota_*M$ is a finitely generated $B_{n+m}(q, P)$-module, and

$$\text{GKdim}_{B_{n+m}(q, P)}(\iota_*M) = \text{GKdim}_{B_n(q, P'')}(M) + m.$$ 

In particular, $M$ is holonomic if and only if $\iota_*M$ is holonomic.

Proof. (1) We use same notations as in Example 3.14. Let $M$ be a finitely generated left $A_m$-module, $\mathcal{F}(M) = \{\mathcal{F}_k(M)\}_{k \geq 0}$ a good filtration of $M$ with respect to the filtration $\mathcal{F}(A_m)$ of $A_m$ defined in Example 1.4. The $K$-subalgebra $C$ of $A_{n+m}$ generated by $x^1, \cdots, x^m, \partial_1, \cdots, \partial_n$ is a filtered $K$-algebra with the filtration $\mathcal{F}(C) = \{\mathcal{F}_k(C)\}$ defined by $\mathcal{F}_0(C) = K$, $\mathcal{F}_1(C) = K1 + Kx^1 + \cdots + Kx^m + K\partial_1 + \cdots + K\partial_n$ and $\mathcal{F}_k(C) = \mathcal{F}_1(C)^k$ ($k \geq 1$). Then, it is clear that $A_{n+m} = C \otimes A_m$ as a $K$-vector space, and that

$$\mathcal{F}_k(A_{n+m}) = \sum_{k' + k'' = k} \mathcal{F}_{k'}(C) \otimes \mathcal{F}_{k''}(A_m) \quad (k \geq 0).$$

Since the left $C$-module $K[X]$ is generated by the single element $1$, the subspaces

$$\mathcal{F}_k(K[X]) = \mathcal{F}_k(C) \cdot 1 \quad (k \geq 0)$$

form a good filtration of $K[X]$ with respect to $\mathcal{F}(C)$. By the same way as in Example 1.8, one sees that $\text{GKdim}_C(K[X]) = n$. 

First, we have to show that $\pi^* M$ is finitely generated. As seen in Example 3.14(1), $\pi^* M$ can be identified with $K[X] \otimes M$. If $M$ is generated by $u_1, \cdots, u_s$, then $K[X] \otimes M$ is generated by $1 \otimes u_1, \cdots, 1 \otimes u_s$.

Next, we will consider the Gelfand-Kirillov dimension of $\pi^* M = K[X] \otimes M$. Since the $K$-linear automorphisms $\psi_j$ and $\psi'_j$ (defined in Example 3.14(1)) preserve the filtration $F(C)$, the $K$-vector spaces $F_k(\pi^* M) = \sum_{k' + k'' = k} F_{k'}(K[X]) \otimes F_{k''}(M)$ form a good filtration of the $A_{n+m}$-module $\pi^* M$ with respect to $F(A_{n+m})$. By the same way as in the proof of [5, Ch.3, Thm.4.1(1)] we have

$$\text{GKdim}_{A_{n+m}}(\pi^* M) = \text{GKdim}_{C}(K[X]) + \text{GKdim}_{A_{m}}(M) = n + \text{GKdim}_{A_{m}}(M).$$

Therefore (1) is proved.

In the similar fashion we can conclude (2)-(4). \qed

**COROLLARY 3.16.** Let notations be as in Example 3.14.

(1) If $M$ is a holonomic $A_{n}(q, P')$-module, then $\pi^* M$ is holonomic as a left $A_{n+m}(q, P)$-module.

(2) If $M$ is a holonomic $A_{n}(q, P'')$-module, then $\iota^* M$ is holonomic as a left $A_{n+m}(q, P)$-module.

**Proof.** By the Lemma 3.15, it suffices to show that

$$\pi^* M/T(\pi^* M) \cong \pi^* (M/T(M)) \quad (\text{as left } A_{n+m}-\text{modules}).$$

But, one easily sees that $T(K[X] \otimes M) = K[X] \otimes T(M)$, which shows the desired isomorphism.

(2) Similar to (1). \qed

**LEMMA 3.17.** Let $n$, $m$ and $r$ be nonnegative integers, $P = (p_{ij})_{1 \leq i, j \leq n}$, $P' = (p'_{ij})_{1 \leq i, j \leq m}$ and $P'' = (p''_{ij})_{1 \leq i, j \leq r}$ matrices as in Example 1.1. Given two DG-algebra morphisms $F : \Omega(q, P') \to \Omega(q, P)$ and $G : \Omega(q, P'') \to \Omega(q, P')$.

(1) For a left $A_{r}(q, P'')$-module $M$,

$$(F \circ G)^*(M) \cong (F^* \circ G^*)(M) \quad (\text{as left } A_{n}(q, P)-\text{modules}),$$

(2) For a left $A_{n}(q, P)$-module $M$,

$$(F \circ G)_*(M) \cong (G_* \circ F_*)(M) \quad (\text{as left } A_{r}(q, P'')-\text{modules}).$$
Proof. We denote by $K[T]$ the quantum affine space $K_{q,P,T}[T]$ with the variables $t^1, \ldots, t^r$.

(1) By definition, $(F \circ G)^*(M) = K[X] \otimes_{K[T]} M$ and $F^*(G^*(M)) = K[X] \otimes_{K[Y]} (K[Y] \otimes_{K[T]} M)$. Thus, there exists a $K[X]$-module isomorphism

$$F^*(G^*(M)) \rightarrow (F \circ G)^*(M), \quad f \otimes g \otimes u \mapsto fF(g) \otimes u,$$

where $f \in K[X], g \in K[Y], u \in M$. Therefore it remains to show that this isomorphism is an $A_n(q,P)$-module morphism. But, using the equation (3.4), one can verify this.

(2) By the definition of the direct images, it suffices that $D_{T \rightarrow X} \cong D_{T \rightarrow Y} \otimes_{A_m} D_{Y \rightarrow X}$ (see the paragraph before Definition 3.14). This follows from Part (1).

\[\square\]

4 Kashiwara’s Theorem for Quantum Weyl Algebras

In this section we consider an analogue of Kashiwara’s theorem for quantum Weyl algebras. See [5, Cor.17.3.2; 4, Thm.V.3.1.6] for Kashiwara’s theorem for the Weyl algebras in characteristic zero. We deal with the category of $B_{n+m}(q,P)$-modules instead of the category of $A_n(q,P)$-modules.

Throughout this section, we use the notations in Example 3.14.

Let $M$ be a left $B_{n+m}(q,P)$-module. Following classical notation, we put

$$\Gamma_{[H]}(M) = \{m \in M \mid (y^j)^s m = 0 \ (j = 1, \ldots, m) \text{ for some } s \in \mathbb{N}\}.$$

Note that, if $u \in M$ with $(y^j)^s u = 0$ for some $s \in \mathbb{N}$, then

$$(y^j)^{s+1}(\widehat{\partial}_j u) = q^{-2(s+1)}(\widehat{\partial}_j(y^j)^{s+1} - [s+1](y^j)^s)u = 0.$$

In classical case, $H$ denotes the hyperplane $\{y^1 = \cdots = y^m = 0\}$. From the above fact and the commutation relations between the generators of $B_{n+m}(q,P)$ described in Remark 1.6, $\Gamma_{[H]}(M)$ is a $B_{n+m}(q,P)$-submodule of $M$. We say that $M$ is supported by $H$ if $M = \Gamma_{[H]}(M)$. Put

$$M_0 = \{u \in M \mid y^j u = 0 \ (j = 1, \ldots, m)\},$$

which is a $B_n(q,P^\prime)$-submodule of $\Gamma_{[H]}(M)$.

Let us consider the case $m = 1$. We write $y$ for $y^1$, $\widehat{\partial}'$ for $\widehat{\partial}_1$, simply.
LEMMA 4.1. Let $M$ be a left $B_{n+1}(q, P)$-module.

1. $\iota_*(M_0) \cong \Gamma([H])(M)$ as left $B_{n+1}(q, P)$-modules.

2. $y\Gamma([H])(M) = \Gamma([H])(M)$.

**Proof.** (Similar to the proof of [5, Thm.17.2.4, Cor.17.2.5].) As seen in Example 3.14(4), $\iota_*(M_0) \cong (B_{n+1}/B_{n+1}\gamma) \otimes_A M_0$. There is a $B_{n+1}$-module morphism $\Theta : (B_{n+1}/B_{n+1}\gamma) \otimes_A M_0 \to \Gamma([H])(M)$ such that

$$\Theta(D \otimes u) = Du \quad (D \in B_{n+1}, u \in M_0).$$

In order to prove the surjectivity of $\Theta$, we have to show that $\Gamma([H])(M) = B_{n+1}M_0$. Let $u \in \Gamma([H])(M)$ with $y^s u = 0$ for some $s \geq 0$. We prove by induction that $u \in B_{n+1}M_0$. Clearly, it is true for $s = 0$. Assume that $s \geq 1$. Since $0 = \hat{\partial}'y^s u = y^{s-1}([s] + q^{2s} y\hat{\partial}')u$, it follows that $([s] + q^{2s} y\hat{\partial}')u \in B_{n+1}M_0$ by induction hypothesis. Again, by induction hypothesis, $yu \in B_{n+1}M_0$, so $q^{2(s-1)}\hat{\partial}'yu \in B_{n+1}M_0$. Therefore $[s - 1]u = ([s] + q^{2s} y\hat{\partial}' - q^{2(s-1)}\hat{\partial}' y)u \in B_{n+1}M_0$, so that $u \in B_{n+1}M_0$.

Next, we claim that

$$B_{n+1}M_0 = \bigoplus_{s \geq 0} \hat{\partial}'^s M_0.$$

Noting that $\mathcal{Z}_i^{-1} M_0 \subset M_0$, one sees that $B_{n+1}M_0 = \sum_{s \geq 0} \hat{\partial}'^s M_0$. Now, assume that

$$u_0 + \hat{\partial}'u_1 + \cdots + \hat{\partial}'^{r-1}u_{r-1} = 0 \quad (u_1, \ldots, u_r \in M_0).$$

Multiplying by $y^r$, we obtain $u_r = 0$ by virtue of the fact that

$$y\hat{\partial}'^{s+1}u = q^{-2(s+1)}[s + 1] \hat{\partial}'^s u \quad (u \in M_0).$$

(4.2) So $u_0 + \hat{\partial}'u_1 + \cdots + \hat{\partial}'^{r-1}u_{r-1} = 0$. By repeating this process, we have $u_0 = \hat{\partial}'^1 u_1 = \cdots = \hat{\partial}'^{s-1} u_{s-1} = 0$. This concludes the claim.

By the claim, in order to show the injectivity of $\Theta$ it is sufficient to prove that $u \in M_0$ with $\hat{\partial}'^s u = 0$ must be zero. Since $y\hat{\partial}'^s u = 0$, it follows from (4.2) that $\hat{\partial}'^{s-1} u = 0$. By continuing this process we get $u = 0$.

Now Part(1) is proved. Since $y(\text{Im } \Theta) = \text{Im } \Theta$ as seen above, Part(2) follows. \hfill $\square$

Denote by $\mathcal{M}^{n+m}$ (resp. $\mathcal{M}^n$) the category of left modules over $B_{n+m}(q, P)$ (resp. $B_n(q, P''')$), and the full subcategory of $\mathcal{M}^{n+m}$ (resp. $\mathcal{M}^n$) consisting of all finitely generated modules is denoted by $\mathcal{M}_{fg}^{n+m}$ (resp. $\mathcal{M}_{fg}^n$). And, $\mathcal{H}^{n+m}$ (resp. $\mathcal{H}^n$) denotes the full subcategory of $\mathcal{M}_{fg}^{n+m}$ (resp. $\mathcal{M}_{fg}^n$) whose objects are holonomic modules. We denote the full subcategory of $\mathcal{M}^{n+m}$ (resp. $\mathcal{M}_{fg}^{n+m}$, $\mathcal{H}^{n+m}$) consisting of $B_{n+m}(q, P)$-modules supported by $H$ by $\mathcal{M}_H^{n+m}$ (resp. $\mathcal{M}_{fg,H}^{n+m}$, $\mathcal{H}_{H}^{n+m}$).
THEOREM 4.3. Let $\iota : \Omega(q,P) \rightarrow \Omega(q,P'')$ be the DG-algebra morphism defined by

$$\iota(x^i) = x^i, \quad \iota(y^j) = 0, \quad \iota(\xi^i) = \xi^i, \quad \iota(\eta^i) = 0 \quad (1 \leq i \leq n, 1 \leq j \leq m).$$

The functor $\iota_*$ defines an equivalence of the category $\mathcal{M}^n$ (resp. $\mathcal{M}_{fg}^n$, $\mathcal{H}^n$) with the category $\mathcal{M}_{H}^{n+m}$ (resp. $\mathcal{M}_{fg,H}^{n+m}$, $\mathcal{H}_{H}^{n+m}$). Furthermore, its inverse is the functor $M \mapsto M_0$.

Proof. Let $\kappa : \mathcal{M}^{n+m} \rightarrow \mathcal{M}^n$ be the functor defined by $\kappa(M) = M_0$ for each left $B_{n+m}$-module $M$. We prove by induction on $m$ that $\iota_*\kappa(M) \cong M$ for any left $B_{n+m}$-module supported by $H$. Clearly, it is true for $m = 0$. For $m = 1$ this is shown in Lemma 4.1(1). Assume that $m \geq 2$. Let $P''$ be the $(n+1) \times (n+1)$ matrix with the $(i,j)$-entry $p_{ij}$. The DG-algebra morphism $\iota_1 : \Omega(q,P) \rightarrow \Omega(q,P'')$ and $\iota_2 : \Omega(q,P'') \rightarrow \Omega(q,P''')$ are morphisms as in Example 3.14(3) such that $\iota = \iota_2 \circ \iota_1$.

We denote by $\kappa_1 : \mathcal{M}^{n+m} \rightarrow \mathcal{M}^{n+1}$ (resp. $\kappa_2 : \mathcal{M}^{n+1} \rightarrow \mathcal{M}^n$) the functor assigning each left $B_{n+m}$-module (resp. $B_{n+1}$-module) $M$ to $M_0$, so that $\kappa = \kappa_2 \circ \kappa_1$. For each left $B_{n+m}$-module $M$ supported by $H$, one sees that $\iota_* \circ \kappa(M) = \iota_1 \circ \iota_2 \circ \kappa_2 \circ \kappa_1(M) \cong \iota_1 \circ \kappa_1(M) \cong M$ by induction hypothesis, since $\kappa_1(M) = \{u \in \kappa_1(M) \mid (y^j)^{s}u = 0$ for some $s\}$. Therefore $\iota_* \circ \kappa$ is isomorphic to the identity functor on $\mathcal{M}_{X}^{n+m}$.

On the other hand, one easily sees that, for each left $B_{m}$-module $N$,

$$\kappa \circ \iota_* (N) \cong ((B_m/\sum_{j=1}^{m} B_my^j) \otimes N) \cong 1 \otimes N \cong N,$$

so the functor $\kappa \circ \iota_*$ is isomorphic to the identity functor on $\mathcal{M}^n$.

Next, we will prove that $M \in \mathcal{M}^{n+m}_H$ is a finitely generated $B_{n+m}$-module if and only if $M_0$ is a finitely generated $B_{n}$-module. Since $\Gamma_{H}^\ast(M) \cong \iota_* (M_0)$ by Lemma 4.1, if $M_0$ is finitely generated, then $\Gamma_{H}^\ast(M)$ is finitely generated by Lemma 3.15(4). Assume that $\Gamma_{H}^\ast(M)$ is finitely generated. It suffices to show that $M_0$ is a noetherian $B_{n}$-module. Let $N_1 \subset N_2 \subset \cdots \subset M_0$ be an ascending chain of $B_{n}$-submodules of $M_0$. Then $\iota_*(N_1) \subset \iota_*(N_2) \subset \cdots \subset \iota_*(M_0)$ be an ascending chain of $B_{n+m}$-submodules of $\iota_*(M_0) = \Gamma_{H}^\ast(M)$, so that there is an integer $s \in \mathcal{N}$ with $\iota_*(N_s) = \iota_*(N_{s}')$ for $s' \geq s$. Then $N_s = N_{s'}$ for $s' \geq s$, since $\kappa \circ \iota_*(N_l) = 1 \otimes N_l$ for each $l$. We conclude that the functor $\iota_*$ defines an equivalence of $\mathcal{M}_{fg,H}^{n}$ with $\mathcal{M}_{H}^{n+m}$.

Finally, in order to prove that the functor $\iota_*$ defines a equivalence of $\mathcal{M}_{H}^{n}$ with $\mathcal{H}_{H}^{n+m}$ we must show that $N \in \mathcal{M}_{H}^{n}$ is holonomic if and only if $\iota_* N$ is holonomic. But, this is proved in Lemma 3.15(4). \qed
5 Preservation of Holonomicity

We continue to use the notations in Example 3.14.

**Lemma 5.1.** Let $\pi$ and $\iota$ be the DG-algebra morphisms in Example 3.14. If a left $B_{n+m}(q, P)$-module $M$ is holonomic, then both the left $B_m(q, P')$-module $\pi_*M$ and the left $B_n(q, P'')$-module $\iota^*M$ are so.

**Proof.** From Example 3.14(2) and the fact that $M = z_{-1}^{-1}M$ it follows that

$$\pi_*M \cong M/\sum_{i=1}^n \hat{\partial}_i M.$$  

Note that there exists a $K$-algebra isomorphism $T : B_{n+m}(q, P) \to B_{m+n}(q^{-1}, \tilde{P})$ such that, for $1 \leq i \leq n, 1 \leq j \leq m,$

$$T(x^i) = \hat{\partial}_i, \quad T(y^j) = \hat{\partial}_j, \quad T(\hat{\partial}_i) = -q^{-2}y^i, \quad T(\hat{\partial}_j') = -q^{-2}x^j,$$

where $\tilde{P} = \{\tilde{p}_{ij}\}_{1 \leq i,j \leq m+n}$ is the $(m+n) \times (m+n)$ matrix such that

$$\tilde{p}_{ij} = p_{n+m-i+1,n+m-j+1} (1 \leq i, j \leq n+m),$$

and $B_{m+n}(q^{-1}, \tilde{P})$ is generated by $x^1, \ldots, x^m, y^1, \ldots, y^n, \hat{\partial}_1, \ldots, \hat{\partial}_m, \hat{\partial}_1', \ldots, \hat{\partial}_n'$ and $\tilde{z}_1^{-1}, \ldots, \tilde{z}_m^{-1}, \tilde{z}_1'^{-1}, \ldots, \tilde{z}_n'^{-1}$ subject to the relations in Remark 1.6. Via this isomorphism, $M$ is a left $B_{m+n}(q^{-1}, \tilde{P})$-module:

$$D \cdot u = T^{-1}(D)u \quad (D \in B_{m+n}(q^{-1}, \tilde{P}), u \in M).$$

Then $M$ is holonomic as a left $B_{m+n}(q^{-1}, \tilde{P})$-module, since $M$ is holonomic as a left $B_{m+n}(q, P)$-module. And, $B_m(q, P')$-module $M/\sum_{i=1}^n \hat{\partial}_i M$ is holonomic if and only if $B_m(q^{-1}, P'')$-module $M/\sum_{i=1}^m \hat{\partial}'_i M$ is so, where $P''$ is the $m \times m$ matrix with $(i, j)$-entry $\tilde{P}_{ij}$. But, in Example 3.14(3) it is seen that $\iota^*M \cong M/\sum_{j=1}^m y^j M$, so it suffices to prove that the left $B_n(q, P'')$-module $M/\sum_{j=1}^m y^j M$ is holonomic.

It is clear for $m = 0$. Consider the case $m = 1$. By Theorem 4.3, $M_0$ is holonomic $B_n$-module. Put $\overline{M} = M/\Gamma_{[H]}M$. It follows from Lemma 4.1(2) that

$$M/yM \cong \overline{M}/y\overline{M} \quad \text{(as left $B_n$-modules)}$$

via the morphism $u + yM \mapsto \overline{u} + y\overline{M} (u \in M)$. Therefore we can assume that $\Gamma_{[H]}M = 0$. 


Let $\mathcal{F}(M) = \{ \mathcal{F}_k(M) \}_{k \geq 0}$ be a good filtration of the $B_{n+m}$-module $M$ for the filtration $\mathcal{F}(B_{n+m})$ of $B_{n+m}$, $\mathcal{F}(M/yM)$ the filtration of the $B_n$-module $M/yM$ induced by the filtration $\mathcal{F}(B_{n+m})$. Then $\mathcal{F}_k(M/yM) = (\mathcal{F}_k(M) + yM)/yM \cong \mathcal{F}_k(M)/(\mathcal{F}_k(M) \cap yM)$. Note that $y\mathcal{F}_{k-1}(M) \subset \mathcal{F}_k(M) \cap yM$ and $\mathcal{F}_{k-1}(M) \cong y\mathcal{F}_{k-1}(M)$ by the assumption. Then we have

$$\dim_K \mathcal{F}_k(M/yM) \leq \dim_K \mathcal{F}_k(M) - \dim_K \mathcal{F}_{k-1}(M).$$

By the same way as in the proof of [5, Lemma 18.1.2], one sees that there exist $c_1, c_2 \in \mathbb{Q}$ such that

$$\dim_K \mathcal{F}_k(M/yM) \leq c_1 k^n + c_2 (k + 1)^{n-1}$$

for any sufficiently large $k$. By [15, Thm.3.2.4] we conclude that $M/yM$ is holonomic.

For a general $m \geq 0$, we can prove by induction the desired result. □

**COROLLARY 5.2.** Let $\pi$ and $\iota$ be the DG-algebra morphisms in Example 3.14. If a left $A_{n+m}(q, P)$-module $M$ is holonomic, then both the left $A_n(q, P')$-module $\pi_* M$ and the left $A_n(q, P'')$-module $\iota^* M$ are so.

**Proof.** We have to show that $B_n \otimes_{A_n} (\iota^* M/T(\iota^* M))$ is holonomic in order to prove the assertion for $\iota^*$. We can regard $\iota^* M = M/\sum y^j M$ as seen in Example 3.14(3). It is easy to verify that there exists a canonical left $A_n$-module epimorphism $\iota^*(M/T(M)) \to \iota^* M/T(\iota^* M)$, which implies that

$$\text{GKdim}_{B_n}(B_n \otimes_{A_n} (\iota^* M/T(\iota^* M))) \leq \text{GKdim}_{B_n}(B_n \otimes_{A_n} (\iota^*(M/T(M)))).$$

Since, for any left $A_{n+m}$-module $N$, the left $B_n$-module $B_n \otimes_{A_n} \iota^* N$ can be regarded as a submodule of the left $B_n$-module $\iota^*(B_{n+m} \otimes_{A_{n+m}} M)$ via the morphism

$$z_1^{-\alpha_1} \cdots z_n^{-\alpha_n} \otimes (u + \sum y^j M) \mapsto (z_1^{-\alpha_1} \cdots z_n^{-\alpha_n} \otimes u) + \sum y^j (B_{n+m} \otimes_{A_{n+m}} M),$$

where $u \in N, \alpha_1, \ldots, \alpha_n \geq 0$. Thus, we have

$$\text{GKdim}_{B_n}(B_n \otimes_{A_n} \iota^*(M/T(M))) \leq \text{GKdim}_{B_n}(\iota^*(B_{n+m} \otimes_{A_{n+m}} (M/T(M))).$$

From the above two inequalities,

$$\text{GKdim}_{B_n}(B_n \otimes_{A_n} (\iota^* M/T(\iota^* M))) \leq \text{GKdim}_{B_n}(\iota^*(B_{n+m} \otimes_{A_{n+m}} (M/T(M)).$$

The left-hand side of the inequality is less than or equal to $n$ by the holonomicity of $M$ and Lemma 5.1. This proves the holonomicity of $\iota^* M$.

In the similar fashion we can prove the holonomicity of $\pi_* M$. □
We define basic DG-algebra morphisms including generalizations of the morphisms in example 3.14.

**DEFINITION 5.3.** Assume that $1 \leq r \leq n$. Let $P = (p_{ij})_{1 \leq i,j \leq n}$ be a $n \times n$ matrix as in Example 1.1. Given an $r$-tuple $i = (i_1, \cdots, i_r)$ with $i_1 < \cdots < i_r$, we denote by $P_i$ the $r \times r$ matrix whose $(k,l)$-entry is $p_{i_k,i_l}$. The generators of $\Omega(q, P_i)$ is denoted by $y^1, \cdots, y^r, \eta^1, \cdots, \eta^r$.

1. The DG-algebra morphism $\pi_i : \Omega(q, P_i) \to \Omega(q, P)$ is defined by $\pi_i(y^s) = x^{i_s}$, $\pi_i(\eta^s) = \xi^{i_s}$ ($1 \leq s \leq r$).

2. The DG-algebra morphism $\iota_i : \Omega(q, P) \to \Omega(q, P_i)$ is defined by $\iota_i(x^{i_s}) = y^s$, $\iota_i(\xi^{i_s}) = \eta^s$ ($1 \leq s \leq r$), $\iota_i(x^i) = 0$, $\iota_i(\xi^i) = 0$ ($i \not\in \{i_1, \cdots, i_r\}$).

3. For an $n$-tuple $c = (c_1, \cdots, c_n)$ such that each $c_i \in K \setminus \{0\}$, we define the DG-algebra morphism $m_c : \Omega(q, P) \to \Omega(q, P)$ by $m_c(x^i) = c_i x^i$, $m_c(\xi^i) = c_i \xi^i$ ($1 \leq i \leq n$).

4. Assume that $n = 1$. For $0 \neq c \in K$, we define the DG-algebra morphism $E_c : \Omega(q, P) \to K$ by $E_c(x) = c$, $E_c(\xi) = 0$.

**PROPOSITION 5.4.** Let notations be as in Definition 5.3. The functors $(\pi_i)^*$, $(\pi_i)_*$, $(\iota_i)^*$, $(\iota_i)_*$, $(m_c)^*$, $(m_c)_*$, $(E_c)^*$ and $(E_c)_*$ preserve the holonomicity.

**Proof.** By slightly modifying the proof of Corollary 3.16 and Corollary 5.2, one can prove the statement for $\pi_i^*$, $\pi_i_*$, $\iota_i^*$ and $\iota_i_*$.

For $m_c$, we note that, for a left $A_n(q, P)$-module $M$, $m_c^*(M) \cong M_\alpha$, $m_c_*(M) \cong M_{\alpha^{-1}}$, where $\alpha$ is the $K$-algebra automorphism of $A_n(q, P)$ such that $\alpha(x^i) = c_i^{-1} x^i$, $\alpha(\partial_i) = c_i \partial_i$ ($1 \leq i \leq n$), and $M_\alpha$ (resp. $M_{\alpha^{-1}}$) is the twisted module by $\alpha$ (resp. $\alpha^{-1}$), that is, $M_\alpha = M$ (resp. $M_{\alpha^{-1}} = M$) (as a $K$-vector space), and the left $A_n(q, P)$-module structure is given by $D \cdot u = \alpha(D)u$ (resp. $D \cdot u = \alpha^{-1}(D)u$),
where $D \in A_n(q, P)$, $u \in M_\alpha$ (resp. $M_{\alpha-1}$). Since $\alpha$ preserves the filtration $\mathcal{F}(A_n(q, P))$ (defined in Remark 1.6), both $M_\alpha$ and $M_{\alpha-1}$ have same Gelfand-Kirillov dimension as that of $M$. In particular, $(m_\epsilon)^*$ and $(m_\epsilon)_*$ preserve the holomonicity.

Consider the functors $(E_c)^*$ and $(E_c)_*$. The quantum Weyl algebra $A = A_1$ is the $K$-algebra generated by the elements $x$ and $\partial$ with the relation $\partial x = q^2 x \partial + 1$. One easily sees that, for any left $A$-module $M$,

$$(E_c)^*(M) \cong M/(x - c) \quad \text{(as $K$-vector spaces)},$$

and, for any finite-dimensional $K$-vector space $W$,

$$(E_c)_*(W) \cong (A/A(x - c))^\text{dim } W \quad \text{(as left $A$-modules)}.$$  

Since $A/A(x - c)$ is isomorphic as a $K$-vector space to $K[\partial]$, it is clear that the left $A$-module $A/A(x - c)$ is holonomic. Therefore $(E_c)_*$ preserves the holomonicity.

It remains to show that, if a left $A$-module $M$ is holonomic, then $M/(x - c)M$ is finite-dimensional. We can assume that $M$ is of the form $A u$ with $\text{ann } u = AD$ ($0 \neq D \in A$). Put $\overline{M} = M/(x - c)M$. Clearly $\overline{M} = \sum_{k \geq 0} K \partial^k$.

Write

$$D = \sum_{0 \leq i \leq r} \sum_{0 \leq j \leq s} C_{ij} x^i \partial^j \quad (C_{ij} \in K).$$

Let $l$ be a positive integer. From the commutation relation between $x$ and $\partial$, we have

$$\partial^l D = \sum_{0 \leq i \leq r} q^{2li} C_{is} x^i \partial^{l+s} + \sum_{k=0}^{l+s-1} g_k \partial^k$$

for some $g_1, \ldots, g_{l+s-1} \in K[X]$. Since $0 = \partial^l D u$, it follows that

$$\left(\sum_{0 \leq i \leq r} C_{is} q^{2li} c^i\right) \partial^{l+s} u \in \sum_{k=0}^{l+s-1} K \partial^k u,$$

where ? denotes the projection $M \to \overline{M}$. Since $q^2$ is not a root of unity, there exists a positive integer $l_0$ such that $\sum_{0 \leq i \leq r} C_{is} q^{2li} c^i \neq 0$ for all $l \geq l_0$. Then

$$\partial^l u \in \sum_{k=0}^{l_0+s-1} K \partial^k u \quad (l \geq l_0 + s),$$

which implies that $\text{dim}_K(\overline{M}) \leq l_0 + s \leq \infty$. \qed
Finally we consider the preservation of holonomicity under the inverse and direct images in the simplest case.

From now on, if $P$ is the $n \times n$ matrix whose entries are all 1, we write $\Omega_n$ for $\Omega(q, P)$.

**PROPOSITION 5.5.** Let $F : \Omega_m \rightarrow \Omega_n$ be a DG-algebra morphism. Then $F$ is a composition of the DG-morphisms in Definition 5.3.

**Proof.** It sufficies to show that $F$ is the DG-algebra morphism satisfying that, there exist $c_1, \ldots, c_m \in K$ and nonnegative integers $i_1, \ldots, i_m$ with $i_1 < \cdots < i_m$ such that $F(y^j) = c_jx^j$ ($j = 1, \ldots, m$), or that there exists $l \leq m$ such that $F(y^j) \in K$ and $F(y^j) = 0$ for $j \neq l$.

Fix $1 \leq j \leq m$. Assume that $F(y^j) \neq 0$. We write

$$F(y^j) = \sum_{i_1, \ldots, i_s \geq 0} C_{i_1 \ldots i_s} (x^1)^{i_1} \cdots (x^s)^{i_s}$$

for some $s \leq n$, where $C_{i_1 \ldots i_s} \neq 0$ for some $i_s > 1$. Noting that

$$d((x^1)^{i_1} \cdots (x^n)^{i_n}) = \sum_{k=1}^n q^{-(i_k+\cdots+i_n)}[[i_k]](x^1)^{i_1} \cdots (x^k)^{i_k-1} \cdots (x^n)^{i_n} \xi^k,$$

it follows that

$$F(\eta^j) = \sum_{i_1, \ldots, i_s \geq 0} C_{i_1 \ldots i_s} \sum_{k=1}^s q^{-(i_k+\cdots+i_n)}[[i_k]](x^1)^{i_1} \cdots (x^k)^{i_k-1} \cdots (x^n)^{i_n} \xi^k.\tag{5.6}$$

Since $y^j \eta^j = q^2 \eta^j y^j$, we have $F(y^j)F(\eta^j) = q^2 F(\eta^j)F(y^j)$, so

$$\sum_{k=1}^s \sum_{i_1, \ldots, i_s \geq 0} \sum_{i'_1, \ldots, i'_s \geq 0} C_{i_1 \ldots i_s} C_{i'_1 \ldots i'_s} q^{-\sum_{l=k+1}^s (i'_l - i_l) + \sum_{l=1}^s \sum_{l'=1}^s i'_l r_s}[[i_k]] \\ \times (x^1)^{i_1+i'_1} \cdots (x^k)^{i_k+i'_k-1} \cdots (x^n)^{i_n+i'_n} \xi^k$$

$$= \sum_{k=1}^s \sum_{i_1, \ldots, i_s \geq 0} \sum_{i'_1, \ldots, i'_s \geq 0} C_{i_1 \ldots i_s} C_{i'_1 \ldots i'_s} q^{-\sum_{l=k+1}^s i'_l}[[i_k]] \\ \times (x^1)^{i_1} \cdots (x^k)^{i_k-1} \cdots (x^n)^{i_n} \xi^k \xi^k (x^1)^{i_1} \cdots (x^n)^{i_n}.\tag{5.6}$$
Since $\xi^k(x^l)^i = q^{-i}(x^l)^i + (q^{-3i+1} - q^{-i+1})(x^l)^{i-1}x^k\xi^l$ for $k > l$, the right-hand side of the equation equals

$$\sum_{k=1}^{s} \sum_{k'=k+1}^{s} \sum_{i_1 \geq 0} \sum_{i'_1 \geq 0} C_{i_1\ldots i_s}C'_{i'_1\ldots i'_s} (q^{-2i_k} - 1)[[i'_k]]$$

$$\times q^{2-(2\sum_{i=k+1}^{s}i'_i + \sum_{i=k+1}^{s}i_i + \sum_{i=k+1}^{s}i'_i - \sum_{i=k+1}^{s}i'_i)}$$

$$\times (x^1)^{i_1+i'_1} \ldots (x^s)^{i_s+i'_s}$$

$$\sum_{k=1}^{s} \sum_{i_1 \geq 0} \sum_{i'_1 \geq 0} C_{i_1\ldots i_s}C'_{i'_1\ldots i'_s}[i'_k]$$

$$\times q^{2-(i_s + \sum_{i=k}^{s}i'_i + \sum_{i=k}^{s}i_i + \sum_{i=k}^{s}i'_i - \sum_{i=k}^{s}i'_i)}$$

$$\times (x^1)^{i_1+i'_1} \ldots (x^s)^{i_s+i'_s}.$$

Then, we claim that $F(y^j) \in Kx^s + K$. Put

$$I_s = \max\{i_s | C_{i_1\ldots i_s} \neq 0 \text{ for some } i_1, \ldots, i_{s-1}\}.$$

Choose an $(s - 1)$-tuple $(I_1, \ldots, I_{s-1})$ such that

$$I_1 + \cdots + I_{s-1} = \max\{i_1 + \cdots + i_{s-1} | C_{i_1\ldots i_{s-1}} I_s \neq 0\}.$$

Comparing the coefficients of $(x^1)^{2I_1} \ldots (x^s)^{2I_{s-1}}(x^s)^{2I_s-1}$ in the both sides of the equation (5.6), we get

$$C^2_{I_1\ldots I_s} q^{-(\sum_{i=1}^{s-1}i'_i + i_s I_0)}[[I_s]] = C^2_{I_1\ldots I_s} q^{2-(2I_s + \sum_{i=1}^{s-1}i'_i + i_s I_0)}[[I_s]],$$

which implies that $I_s = 1$. Now, we assume that $F(y^j) \not\in Kx^s + k[x^1, \ldots, x^{s-1}]$. Put

$$s' = \max\{k < s | C_{i_1\ldots i_{s'}} \neq 0 \text{ for some } i_1, \ldots, i_{s-1} \text{ with } i_{s'} \geq 1\},$$

$$I_{s'} = \max\{i_{s'} | C_{i_1\ldots i_{s'-1}} \neq 0 \text{ for some } i_1, \ldots, i_{s-1}, i_{s'+1}, \ldots, i_{s-1}\},$$

and choose an $(s' - 1)$-tuple $(I_1, \ldots, I_{s'-1})$ such that

$$I_1 + \cdots + I_{s'-1} = \max\{i_1 + \cdots + i_{s'-1} | C_{i_1\ldots i_{s'-1}} I_{s'-0} \neq 0\}.$$

Then, by comparing the coefficients of $(x^1)^{2I_1} \ldots (x^{s'})^{2I_{s'-1}}(x^{s'})^2\xi'$ in the both sides of the equation (5.6), we get

$$C^2_{I_1\ldots I_{s'-0-1}}[[I_{s'}]] = C^2_{I_1\ldots I_{s'-0-1}}((q^{-2I_{s'}} - 1)q^2 + [[I_{s'}]]q^{-2I_{s'}}).$$
This implies that \( q^{i+1} = 1 \), which is a contradiction, because \( q^2 \) is not a root of unity. Therefore \( f(y^j) = Kx^s + K[x^1, \cdots, x^{s-1}] \). By repeating same process as above, one can sees that \( F(y^j) \in \sum_{k=1}^s Kx^k + K \). But, since \( F(\eta^j)^2 = 0 \), it must hold that \( F(y^j) \in Kx^s + K \). Furthermore, from the relation \( F(y^j)F(\eta^j) = q^2 F(\eta^j)F(y^j) \), one sees that \( F(y^j) \in Kx^s \) or \( F(y^j) \in K \).

Now, we know that, for each \( 1 \leq j \leq m \), there exists \( \alpha_j \in K \) such that \( F(y^j) = \alpha_j \) or \( F(y^j) = \alpha_j x^{i_j} \) for some \( i_j \). From the commutation relations of \( \Omega_n \) and \( \Omega_m \), the desired result follows.

Combining the proposition with Proposition 5.4 we obtain the following result:

**Theorem 5.7.** Let \( F : \Omega_m \to \Omega_n \) be a DG-algebra morphism. We regard \( F^* \) (resp. \( F_* \)) as a functor from the category of left \( A^q_m \)-modules (resp. \( A^q_n \)-modules) to that of left \( A^q_n \)-modules (resp. \( A^q_m \)-modules). Then, both the functor \( F^* \) and \( F_* \) preserve the holonomicity.

**References**


