On group association schemes with the same parameters

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Abstract

We show that for the split and non-split extensions of $\mathbf{F}_q^2$ by $SL(2,q)$ ($q = 2^e$, $e \geq 3$), the group association schemes have the same parameters but are not isomorphic. For the split and non-split extensions of $\mathbf{F}_q^2$ by the standard Borel subgroup of $SL(2,q)$ ($q = 2^e$, $e \geq 3$), the group association schemes are shown to be isomorphic.

1 Introduction

The following immediately follow from the definition of group association schemes:

(a) The group association schemes $\mathcal{X}(G)$ and $\mathcal{X}(H)$ are isomorphic if G and H are isomorphic as groups,

(b) If the group association schemes $\mathcal{X}(G)$ and $\mathcal{X}(H)$ are isomorphic, then they have the same sets of parameters.

The conclusion of (b) holds if and only if G and H have the same character table ([1, (7.1), pp. 42–43]). The converse of each of the statements (a) and (b) is known to be false. About the converse of (a), some nilpotent groups are known as counterexamples, but no non-nilpotent groups seem to have appeared in the literature. As for (b), the pair of split and non-split extensions of $\mathbf{F}_2^3$ by $SL(3,2)$ found by Yoshiara [4] is so far the unique known counterexample. The construction of counterexamples to the converse of (b) seems rather difficult, because we need more information than parameters of group association schemes to determine whether or not they are isomorphic.

In this note, I give an infinite number of pairs of non-nilpotent counterexamples to the converse of (a), and of non-solvable ones to that of (b).

Main Theorem

(1) *For $q = 2^e \geq 8$, the group association scheme for the split extension of $\mathbf{F}_q^2$ by the standard Borel subgroup of $SL(2, q)$ is isomorphic to the group association scheme for the non-split extension of $\mathbf{F}_q^2$ by the standard Borel subgroup of $SL(2, q)$ although the groups are not isomorphic.*

(2) *For $q = 2^e \geq 8$, the group association scheme for the split extension of $\mathbf{F}_q^2$ by $SL(2, q)$ is not isomorphic to the group association scheme for the non-split extension of $\mathbf{F}_q^2$ by $SL(2, q)$, but they have the same parameters.*

This note is organized as follows. In §2, we will give the definition of a group association scheme $\mathcal{X}(G)$ for a group G and discuss briefly the behavior of the automorphism group of $\mathcal{X}(G)$. The conjugacy classes and the character tables of the extensions in Main Theorem are calculated in §3 using the results of Bell [2]. In §4, we prove Main Theorem (1) via Lemma 4.2 which is available for a more general setting. This gives us counterexamples to the above statement (a) that are not nilpotent. Main Theorem (2) is established by calculating the automorphism groups of the schemes. More precisely, we consider the stabilizer of the identity element in the automorphism group of the association scheme of the split extension (denoted A in §4). The group A acts on the set Δ of some classes corresponding to the 1-dimensional subspaces of the natural module. We may see that A/N induces a Zassenhaus group on Δ with the kernel N of the action A onto Δ. Then A is determined in Preposition 4.4, from which we easily conclude that the schemes of the split and non-split extensions are not isomorphic (Theorem 4.5). To establish the structure of the stabilizer A, we had examined the neighbors of some vertices with respect to various relations by observing the traces of many matrices.

2 The group association scheme

The *group association scheme* $\mathcal{X}(G)$ for a group G with conjugacy classes $\mathcal{C}_0, \ldots, \mathcal{C}_d$ is the pair $\mathcal{X}(G) = (G, \{R(\mathcal{C}_i)\}_{i=0}^d)$ of the set G and the set of relations $\{R(\mathcal{C}_i) \mid i = 0, 1, \ldots, d\}$ defined by $(x, y) \in R(\mathcal{C}_i)$ if and only if $x^{-1}y \in \mathcal{C}_i$. The parameters of the group association scheme $\mathcal{X}(G)$ can be calculated from the character table of G:

$$p_{i,j}^k = a(\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k) = \frac{|\mathcal{C}_i||\mathcal{C}_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_k)}}{\chi(1)},$$

where $\mathrm{Irr}(G)$ is the set of irreducible characters of G, and g_l is a representative of a conjugacy class $\mathcal{C}_l$ for $l = i, j, k$. For a conjugacy class $\mathcal{D}$ of G, the $R(\mathcal{D})$-graph is a graph with vertex set G and edge set $R(\mathcal{D}) = \{(x, y) \in G \times G \mid x^{-1}y \in \mathcal{D}\}$. Two group association schemes $\mathcal{X}(G) = (G, \{R(\mathcal{C}_i)\}_{i=0}^d)$ and $\mathcal{X}(H) = (H, \{R(\mathcal{D}_i)\}_{i=0}^d)$ with the same number of association relations are called *isomorphic* when there is a bijection from G to H which sends each relation $R(\mathcal{C}_i)$ to $R(\mathcal{D}_i)$ $(i = 0, \ldots, d)$ after arranging the ordering of relations. Especially, we call the set of isomorphisms the *full automorphism group* $\mathrm{Aut}(\mathcal{X}(G))$ if $G = H$ and $\mathcal{C}_i = \mathcal{D}_i$ $(i = 0, \ldots, d)$.

Let ρ be the homomorphism from $G \times G$ to $\mathrm{Aut}(\mathcal{X}(G))$ defined by

$$\rho(g,h)x = gxh^{-1} \quad \text{for any } g,h,x \in G.$$

It is easy to verify that the image of ρ lies in $\mathrm{Aut}(\mathcal{X}(G))$, and that

$$\rho(G \times G) \cong G * G,$$

where $G * G$ means the central product of G:

$$G * G := G \times G / \{(z,z) \mid z \in Z(G)\}.$$

Hence the full automorphism group $\mathrm{Aut}(\mathcal{X}(G))$ has a subgroup which is isomorphic to $G * G$. Moreover, denoting the set $\{\mathrm{Inn}(g) := \rho(g,g) \mid g \in G\}$ by $\mathrm{Inn}(G)$, we have $\mathrm{Inn}(G) \cong G/Z(G)$ and $\mathrm{Inn}(G)$ is a subgroup of the stabilizer of the identity 1 of G in the full automorphism group $\mathrm{Aut}(\mathcal{X}(G))$. Note that the group $\mathrm{Aut}(\mathcal{X}(G))$ acts on G transitively since it contains a transitive subgroup $\rho(G \times 1)$ on G.

3 Extensions of $\mathbf{F}_{2^e}^2$ by $SL(2, 2^e)$

Let G be an arbitrary finite group, V a $\mathbf{Z}[G]$-module, and f a function from $G \times G$ to V. Denote E_f by the set $V \times G$ equipped with the product

$$(v,g)(w,h) := (v + g.w + f(g,h),\ gh),$$

where $g.w$ is the action of g on w. Then E_f forms a group if and only if f is a normalized 2-cocycle (i.e., $f(1,g) = 0 = f(g,1)$ and $g.f(h,k) = f(gh,k) - f(g,hk) + f(g,h)$ for all $g,h,k \in G$). This group E_f is called an *extension* of V by G with respect to f.

Theorem 3.1 [2] *Let q be a power of 2, $SL(2,q)$ the special linear group of 2×2 matrices over $\mathbf{F}_q$, the finite field of q elements, and let V be the natural module for $SL(2,q)$.*

If $q \geq 8$, then $H^2(SL(2,q),V) = \{[f_k] \mid k \in \mathbf{F}_q\}$, where

$$f_k|_{U\times U}(\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} k(uw)^{1/4} \\ 0 \end{bmatrix}$$

for $u,w \in \mathbf{F}_q$, U denotes the standard unipotent radical of $SL(2,q)$, and $u^{1/4}$ stands for the element of $\mathbf{F}_q$ whose fourth power equals u.

In this note, we use the following notation:

$q = 2^e$ with $e \geq 3$, δ is a generator of the multiplicative group $\mathbf{F}_q^\times$,
$V \cong \mathbf{F}_q^2$, the natural module for $SL(2,q)$,
$E_k := E_{f_k}$ where f_k denotes the 2-cocycle above,
$U_k := \{\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}\right)_k \mid \alpha,\beta,u \in \mathbf{F}_q\} \leq E_k$,
$T_k := \left\langle (\mathbf{0}, \mathrm{diag}(\delta,\delta^{-1}))_k = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{bmatrix}\right)_k \right\rangle \leq E_k$, and
the matrix part of an element g_k of E_k will be denoted by $\overline{g_k}$:

$$E_k \ni g_k = (\mathbf{x}, M)_k \mapsto \overline{g_k} = M \in SL(2,q).$$

The map $(\mathbf{x}, M)_1 \mapsto (k\mathbf{x}, M)_k$ gives an isomorphism of E_1 with E_k for $0 \neq k \in \mathbf{F}_q$, while $E_0 \not\cong E_1$, because E_0/V splits but E_1/V does not.

We partially identify the symbol V with the subgroup $\{(v, I)_k \mid v \in V\}$ of E_k.

To calculate the conjugacy classes of E_k, we use the following two elementary observations.

Lemma 3.2 *If $g_1, \ldots, g_r$ are complete representatives of conjugacy classes of $SL(2, q)$, then $Vg_1, \ldots, Vg_r$ are complete representatives of E_k-conjugacy classes in the cosets $V\backslash E_k$.*

Lemma 3.3 *For each $e_k = (v, g)_k \in E_k$, the following hold:*

(1) *The V-conjugacy class of e_k coincides with $[V, e_k]e_k$.*

(2) *If $(w, h)_k(v, g)_k(w, h)_k^{-1} = (v', g)_k$, then h is contained in the centralizer of g in $SL(2, q)$.*

(3) *If e_k acts on $V - \{1\}$ fixed point freely, then Ve_k forms a single V-conjugacy class.*

Lemma 3.4 *The conjugacy classes of E_k are those given in Table 1, where s is a generator of a Singer cycle of $SL(2, q)$ and δ is a generator of the multiplicative group $\mathbf{F}_q^\times$ of the field $\mathbf{F}_q$.*

Table 1: Conjugacy classes of E_k $(k = 0, 1)$

Class name	*Rep.*	*Size*	*Order*
$1(k)$	1_k	1	1
$\mathcal{V}^\sharp(k)$	$\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1&0\\0&1\end{bmatrix}\right)_k$	$(q+1)(q-1)$	2
$\mathcal{U}_\beta(k)$ $(\beta \in \mathbf{F}_q)$	$\left(\begin{bmatrix}0\\\beta\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}\right)_k$	$q(q+1)(q-1)$	$\begin{cases}2 \text{ if } \beta = k\\ 4 \text{ otherwise}\end{cases}$
$\mathcal{S}_j(k)$ $(j = 1, \ldots, q/2)$	$\left(\begin{bmatrix}0\\0\end{bmatrix}, s\right)_k^j$	$q^3(q-1)$	$\frac{q+1}{\gcd(q+1,j)}$
$\mathcal{T}_i(k)$ $(i = 1, \ldots, (q-2)/2)$	$\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\delta&0\\0&\delta^{-1}\end{bmatrix}\right)_k^i$	$q^3(q+1)$	$\frac{q-1}{\gcd(q-1,i)}$

Proposition 3.5 *Let η (resp. ξ) be a primitive $(q-1)$-st (resp. $(q+1)$-st) root of unity in the field $\mathbf{C}$ of complex numbers. For the additive character ψ of $(\mathbf{F}_q, +)$ defined by $\psi : \mathbf{F}_q \ni \alpha \mapsto (-1)^{Tr_{\mathbf{F}_q/\mathbf{F}_2}(\alpha)} \in \mathbf{C}^\times$, set*

$$K(\psi; \beta, b) := \sum_{u \in \mathbf{F}_q^\times} \psi(u^{-\frac{1}{2}}\beta + bu)$$

Then irreducible characters of E_k are those given in Table 2. They do not depend on the choice of k.

Table 2: The irreducible characters of E_k

Class name	$1(k)$	$\mathcal{V}^\sharp(k)$	$\mathcal{T}_i(k)$ $(i=1,\ldots,(q-2)/2)$	$\mathcal{S}_j(k)$ $(j=1,\ldots,q/2)$	$\mathcal{U}_\beta(k)$ $(\beta\in\mathbf{F}_q)$
Size	1	q^2-1	$q^3(q+1)$	$q^3(q-1)$	$q(q^2-1)$
	1	1	1	1	1
	q	q	1	-1	0
$(m=1,\ldots,(q-2)/2)$	$q+1$	$q+1$	$\eta^{mi}+\eta^{-mi}$	0	1
$(n=1,\ldots,q/2)$	$q-1$	$q-1$	0	$-(\xi^{nj}+\xi^{-nj})$	-1
$\psi_{1b}\!\uparrow^{E_k}$ $(b\in\mathbf{F}_q)$	q^2-1	-1	0	0	$K(\psi;\beta,b)$

Table 3: Conjugacy classes of U_k

Class name	Rep. x_k	U_k-orbit of x_k	Size
$\mathcal{C}^I_\alpha(k)$ $(\alpha\in\mathbf{F}_q)$	$\left(\begin{bmatrix}\alpha\\0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)_k$	$\left\{\left(\begin{bmatrix}\alpha\\0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)_k\right\}$	1
$\mathcal{C}^{II}_{\beta,w}(k)$ $(\beta,w\in\mathbf{F}_q,$ $(\beta,w)\neq(0,0))$	$\left(\begin{bmatrix}0\\\beta\end{bmatrix},\begin{bmatrix}1&w\\0&1\end{bmatrix}\right)_k$	$\left\{\left(\begin{bmatrix}\alpha\\\beta\end{bmatrix},\begin{bmatrix}1&w\\0&1\end{bmatrix}\right)_k \mid \alpha\in\mathbf{F}_q\right\}$	q

4 Sketch to prove Main Theorem

We see $U_0\not\cong U_1$, as U_0/V splits but U_1/V does not. For $0\neq k\in\mathbf{F}_q$, let τ_k be the map from U_1 to U_k defined by

$$\tau_k\left(\left(\begin{bmatrix}\alpha\\\beta\end{bmatrix},\begin{bmatrix}1&u\\0&1\end{bmatrix}\right)_1\right)=\left(k\begin{bmatrix}\alpha\\\beta\end{bmatrix},\begin{bmatrix}1&u\\0&1\end{bmatrix}\right)_k.$$

Then τ_k becomes an isomorphism.

It is straightforward to calculate the conjugacy classes of U_k for an arbitrary $k\in\mathbf{F}_q$. The result is given in Table 3.

Lemma 4.1 *We have $\mathcal{X}(U_1)\cong\mathcal{X}(U_0)$ via the automorphism ζ sending X_1 to X_0.*

Lemma 4.2 *Let G_1,G_2,H_1,H_2 be groups satisfying the following conditions* (i)-(iii)*.*

(i) *H_i acts fixed point freely on G_i for $i=1,2$.*
(ii) *$\mathcal{X}(G_1)$ is isomorphic to $\mathcal{X}(G_2)$ via an isomorphism ζ.*
(iii) *H_1 is isomorphic to H_2 via an isomorphism φ.*

We naturally embed H_i and G_i in the semi-direct product $A_i:=G_i{:}H_i$. Assume

$$\varphi(h)\zeta(x)\varphi(h)^{-1}\in\zeta(h\mathcal{C}h^{-1})$$

for every x in each conjugacy class $\mathcal{C}$ of G_1 and every h of H_1. Then $\mathcal{X}(A_1)$ and $\mathcal{X}(A_2)$ are isomorphic via the map

$$\zeta\circ\varphi: A_1\ni gh\mapsto\zeta(g)\varphi(h)\in A_2.$$

Corollary 4.3 *We have $\mathcal{X}(U_0{:}T_0) \cong \mathcal{X}(U_1{:}T_1)$. That is to say, Main Theorem (1) holds.*

In the following, we will show that for $q = 2^e \geq 8$, the group association scheme $\mathcal{X}(E_0)$ for the split extension E_0 of $\mathbf{F}_q^2$ by $SL(2,q)$ is not isomorphic to the group association scheme $\mathcal{X}(E_k)$ for the non-split extension $E_k \cong E_1$ ($k \in \mathbf{F}_q^\times$) of $\mathbf{F}_q^2$ by $SL(2,q)$. We will prove it by calculating the full automorphism group $\mathrm{Aut}(\mathcal{X}(E_0))$.

Proposition 4.4 $A = \mathrm{Inn}(E_0) \times \langle \iota \rangle \cong E_0 \times 2$.

Theorem 4.5 $\mathrm{Aut}(\mathcal{X}(E_0)) \cong (E_0 \times E_0) : 2$. *Moreover, $\mathcal{X}(E_0)$ is not isomorphic to $\mathcal{X}(E_1)$.*

Proof of Proposition 4.4. We can see that A/N is a Zassenhaus group on Δ. As $q = |\Delta| - 1$ is a power of 2, A/N is a group isomorphic to $SL(2,q)$ or the Suzuki group ${}^2B_2(q^{1/2})$ of order $q(q+1)(q^{1/2}-1)$. On the other hand, we have

$$\begin{aligned} A/N &\geq \mathrm{Inn}(E_0)N/N \cong \mathrm{Inn}(E_0)/\mathrm{Inn}(E_0) \cap N \\ &= \mathrm{Inn}(E_0)/\{\mathrm{Inn}(\mathbf{v}) \mid \mathbf{v} \in V\} \\ &\cong E_0/V \cong SL(2,q). \end{aligned}$$

Thus A/N has to be $SL(2,q)$. Since $A \geq \mathrm{Inn}(E_0)$ and $[\mathrm{Inn}(E_0), \iota] = 1$, we have $A = \mathrm{Inn}(E_0) \times \langle \iota \rangle \cong E_0 \times 2$. □

Proof of Theorem 4.5. ¿From Proposition 4.4, we have $A = \mathrm{Aut}(\mathcal{X}(E_0))_1 \cong E_0 \times 2$, and hence, $|\mathrm{Aut}(\mathcal{X}(E_0))| = |E_0||A| = |(E_0 \times E_0){:}2|$. As we saw in §1, $\mathrm{Aut}(\mathcal{X}(E_0))$ has a subgroup $\rho(E_0 \times E_0)$ isomorphic to $E_0 \times E_0/\{(z,z) \mid z \in Z(E_0)\} \cong E_0 \times E_0$. Hence $\mathrm{Aut}(\mathcal{X}(E_0)) \cong (E_0 \times E_0){:}\langle \iota \rangle$ since $\iota \notin \{\rho(g,h) \mid g,h \in E_0\}$ and $\iota\rho(g,h)\iota^{-1} = \rho(h,g)$.

If $\mathcal{X}(E_0)$ is isomorphic to $\mathcal{X}(E_1)$, then $\mathrm{Aut}(\mathcal{X}(E_0))$ is also isomorphic to $\mathrm{Aut}(\mathcal{X}(E_1))$. Thus $\mathrm{Aut}(\mathcal{X}(E_0))$ has a subgroup isomorphic to $\mathrm{Inn}(E_1) \cong E_1$. This contradicts the structure of $\mathrm{Aut}(\mathcal{X}(E_0)) \cong (E_0 \times E_0){:}2$. Hence $\mathcal{X}(E_0)$ is not isomorphic to $\mathcal{X}(E_1)$. □

As we saw in §3, E_0 and E_1 have the same character table. Thus the pairs $E_0 = V{:}SL(2,q)$ and $E_1 = V \cdot SL(2,q)$ for $q = 2^e \geq 8$ form a family of counterexamples to the converse of the statement (b) in §1.

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