SYMMETRY OF ALMOST HEREDITARY RINGS *

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In [6] an almost N-projective module is defined as a generalization of a N-projective module to characterize the lifting property. This modules is further studied in the succeeding papers [7], [8]. And in [3] an almost N-injective module is defined as a concept dual to an almost N-projective module, which is a generalization of a N-injective module. Also in [4], M.Harada and Y.Baba showed the following relations between lifting (resp. extending) modules and almost N-projective (resp. almost N-injective) modules:

Let $\{M_\alpha\}_I$ be a set of modules with each endomorphism ring local.

(a) Assume that M_α is hollow and cyclic for any α. Then the following are equivalent.

- (i) $\bigoplus_I M_\alpha$ is a lifting module
- (ii) M_α is almost M_β-projective for any $\alpha \neq \beta$ and $\{M_\alpha\}_I$ is locally semi-T-nilpotent

(b) Assume that M_α is uniform for any α and $\{M_\alpha\}_I$ is locally semi-T-nilpotent. Then the following are equivalent.

- (i) $\bigoplus_I M_\alpha$ is an extending module
- (ii) M_α is almost M_β-injective for any $\alpha \neq \beta$

And in [10] M. Harada called a module M to be *almost projective* (resp. *almost injective*) if M is almost N-projective (resp. almost N-injective) for any finitely generated module N. We see that semisimple rings, serial rings, QF-rings and H-rings are well-characterized by the properties of an almost projective module and almost injective module in [10], [11]:

* The detail version of this note will be submitted for publication elsewhere.

Let R be an artinian ring. Then the following conditions $(a) - (d)$ hold.

(a) The following are equivalent:

(i) R is a semisimple ring;

(ii) every almost projective right R-module is injective;

(iii) every almost injective right R-module is projective.

(b) The following are equivalent:

(i) R is a serial ring;

(ii) every almost projective right R-module is almost injective;

(iii) every almost injective right R-module is almost projective.

(c) Assume that $eJ \neq 0$ for any primitive idempotent e, where J is the Jacobson radical of R. Then the following are equivalent:

(i) R is a QF-ring;

(ii) $R/Socle(R)_R$ is almost projective;

(iii) J_R is almost injective, but eJ^2 is either zero or never almost injective for any primitive idempotent e.

(d) The following are equivalent:

(i) R is right Harada ring;

(ii) every indecomposable injective right R-module is almost projective;

(iii) every finitely generated projective left R-module is almost injective.

Using this remarkable module, in [9] he defined a *right almost hereditary ring* R, i.e., R is an artinian ring with J_R almost projective, where J is the Jacobson radical of R. On the other hand, it is well known that an artinian hereditary ring R is characterized by the following equivalent conditions:

(1) J_R is projective;

(2) ${}_RJ$ is projective;

(3) $E/Socle(E)$ is injective for any injective right R-module E;

(4) $E/Socle(E)$ is injective for any injective left R-module E.

Therefore a right almost hereditary ring is a generalization of an artinian hereditary ring. In this paper, first we characterize a right almost hereditary ring using left ideals in section 2 (we note that M. Harada already gave a structure theorem of it using right ideals in [9]). Further we consider the following generalized condition of (3):

$(\#)_r$ A factor module of E by its socle is a direct sum of an injective module and finitely generated almost injective modules for any injective right R-module E (not necessarily finitely generated).

Symmetrically we consider the left version $(\#)_l$. And we show that a ring R is a right almost hereditary ring if and only if it satisfies $(\#)_l$ using a characterization of a right almost hereditary ring given by left ideals. But M. Harada already showed that a right almost hereditary ring is not always a left almost hereditary ring in [9, p801]. That is, the equivalences (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (3) are generalized. But the other equivalences are not generalized.

In [9] he further considered the following stronger conditions than one of an almost hereditary ring :

$(*)_r$ Every submodule of a finitely generated projective right R-module is almost projective.

$(**)_r$ The Jacobson radical of M is almost projective for any finitely generated almost projective right R-module M;

$(***)_r$ every submodule of a finitely generated almost projective right R-module is also almost projective.

In this paper we call an artinian ring R a *right strongly almost hereditary ring* (abbreviated *right SAH ring*) if R satisfies $(*)_r$. On the other hand, an artinian hereditary ring is also characterized by the following equivalent conditions:

(a) Every submodule of a projective right R-module is also projective;

(b) every submodule of a projective left R-module is also projective;

(c) every factor module of an injective right R-module is also injective;

(d) every factor module of an injective left R-module is also injective.

In section 3 we consider the following generalized condition of (c):

$(*^{\#})_r$ Every factor module of an injective right R-module is a direct sum of an injective module and finitely generated almost injective modules.

Similarly we define $(*^{\#})_l$ for left R-modules. The aim of Section 3 is to show that an artinian ring R is right SAH if and only if R satisfies $(*^{\#})_l$. But we see that the equivalence between a right SAH ring and an artinian ring which satisfies $(*^{\#})_r$ does not hold in general.

In [9] M. Harada also showed that an artinian ring R satisfies $(**)_r$ iff it satisfies $(***)_r$. In section 4 we consider the following generalized conditions of (c):

$(**^{\#})_r$ $M/Socle(M)$ is a direct sum of an injective module and finitely generated almost injective modules for any injective or finitely generated almost injective right R-module M;

$(\ast\ast\ast^{\#})_r$ every factor module of an injective or finitely generated almost injective right R-module is a direct sum of an injective module and finitely generated almost injective modules.

We also consider $(\ast\ast^{\#})_l$ and $(\ast\ast\ast^{\#})_l$ for left R-modules. The aim of Section 4 is to show that an artinian ring R satisfies $(\ast\ast)_r$ if and only if R satisfies $(\ast\ast^{\#})_l$ if and only if R satisfies $(\ast\ast\ast^{\#})_l$. But we see that the equivalence between the two conditions $(\ast\ast)_r$ and $(\ast\ast^{\#})_r$ does not hold in general.

§1 Preliminaries

In this paper, we always assume that every ring is a basic artinian ring with identity and every module is unitary. Let R be a ring and let $P(R) = \{e_i\}_{i=1}^n$ be a complete set of pairwise orthogonal primitive idempotents in R. We denote the *Jacobson radical*, an *injective hull* and the *composition length* of a module M by $J(M)$, $E(M)$ and $|M|$, respectively. Especially, we put $J := J(R_R)$. For a module M we denote the *socle* of M by $S(M)$ and the *k-th socle* of M by $S_k(M)$ (i.e., $S_k(M)$ is a submodule of M defined by $S_k(M)/S_{k-1}(M) = S(M/S_{k-1}(M))$ inductively).

Let M and N be modules. M is called *almost N-projective* (resp. *almost N-injective*) if for any homomorphism $\phi : M \to L$ (resp. $\phi' : L \to M$) and any epimorphism $\pi : N \to L$ (resp. monomorphism $\iota : L \to N$) either there exists a homomorphism $\tilde{\phi} : M \to N$ (resp. $\tilde{\phi}' : N \to M$) such that $\phi = \pi\tilde{\phi}$ (resp. $\phi' = \tilde{\phi}'\iota$) or there exist a nonzero direct summand N' of N and a homomorphism $\theta : N' \to M$ (resp. $\theta' : M \to N'$) such that $\phi\theta = \pi i$ (resp. $\theta'\phi' = p\iota$), where i is an inclusion of N' in N (resp. p is a projection on N' of N).

A ring R is called *right* (resp. *left*) *hereditary* if every submodule of a projective right (resp. left) R-module is also projective. It is well known that a perfect or neotherian ring is right hereditary iff it is left hereditary (see, for instance, [13, Chapter 9]). So we call a right hereditary ring a *hereditary* ring since rings are artinian in this paper. Further an artinian ring R is hereditary iff J_R is projective (see, for instance, [1, 18. Exercises 10 (2)]). Furthermore an artinian ring R is hereditary iff $E/S(E)$ is injective for any injective right R-module E. We also see that R is hereditary iff E/A is injective for any submodule A of an injective module E by [1, 18. Exercises 10 (1)].

Further M is called *almost projective* (resp. *almost injective*) if M is always almost N-projective (resp. almost N-injective) for any finitely generated R-module N. The following is an important characterization of an almost projective module given by M. Harada.

Lemma 1 *([10, Corollary $1^{\#}$]). Suppose that M is an indecomposable finitely generated left R-module. Then M is almost injective but not injective if and only if there exist an indecomposable injective left R-module E and a positive integer k such that $M \cong J^k E$ and $J^i E$ is projective for any $i = 0, \cdots, k-1$.*

And we call an artinian ring R a *right almost hereditary ring* if J is almost projective as a right R-module. By [10, Theorem 1] this definition is equivalent to the condition: $J(P)$ is almost projective for any finitely generated projective right R-module P.

A module is called *uniserial* if its lattice of submodules is a finite chain, i.e., any two submodules are comparable. An artinian ring R is called a *right* *serial* (resp. *co-serial*) *ring* if every indecomposable projective (resp. injective) right R-module is uniserial. And we call a ring R a *serial ring* if R is a right and left serial ring. Let $f_1, f_2, \cdots, f_n$ be primitive idempotents in a serial ring R. Then a sequence $\{f_1R, f_2R, \cdots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \cdots, Rf_n\}$) of indecomposable projective right (resp. left) R-modules is called a *Kupisch series* if $f_jJ/f_jJ^2 \cong f_{j+1}R/f_{j+1}J$ (resp. $Jf_j/J^2f_j \cong Rf_{j+1}/Jf_{j+1}$) holds for any $j = 1, \cdots, n-1$. Further $\{f_1R, f_2R, \cdots, f_nR\}$ (resp. $\{Rf_1, Rf_2, \cdots, Rf_n\}$) is called a *cyclic Kupisch series* if it is a Kupisch series and $f_nJ/f_nJ^2 \cong f_1R/f_1J$ (resp. $Jf_n/J^2f_n \cong Rf_1/Jf_1$) holds. Let R be a serial ring with a Kupisch series $\{f_1R, f_2R, \cdots, f_nR\}$. If $f_nJ = 0$ and $P(R) = \{f_1, \cdots, f_n\}$, then R is called a serial ring *in the first category*. And if $\{f_1R, f_2R, \cdots, f_nR\}$ is a cyclic Kupisch series and $P(R) = \{f_1, \cdots, f_n\}$, then R is called a serial ring *in the second category*.

§2 A structure theorem for an almost hereditary ring

The following is a structure theorem for a right almost hereditary ring given by M. Harada.

Theorem 2 *([9, Theorem 1]).* *A ring is right almost hereditary if and only if it is a direct sum of the following rings:*

(i) *Hereditary rings;*

(ii) *serial rings;*

(iii) *rings R with $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$ such that, for each $l = 1, \cdots, k$ we put $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$ and*

$p_l := |f_1^{(l)}R_R|$, *the following four conditions hold for any* $l = 1, \cdots, k$ *and* $s = 1, \cdots, m$,

(a) S_lRS_l *is a serial ring in the first category with* $\{f_1^{(l)}RS_l,\ f_2^{(l)}RS_l,\ \cdots,\ f_{n_l}^{(l)}RS_l\}$ *a Kupisch series of right* S_lRS_l*-modules,*

(b) $S_lR(1-S_l) = 0$, $(h_1 + \cdots + h_m)R(f_1^{(l)} + \cdots + f_{p_l-1}^{(l)}) \neq 0$ *and* $(h_1 + \cdots + h_m)R(f_{p_l}^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$,

(c) $(h_sJ/h_sJ^2)f_j^{(l)} = \overline{0}$ *for any* $j \geq 2$,

we let α_l *be a positive integer such that* $f_1^{(l)}R/f_1^{(l)}J^j$ *is injective for any* $j\ (\geq \alpha_l + 1)$ *but* $f_1^{(l)}R/f_1^{(l)}J^{\alpha_l}$ *is not injective (see Lemma* **??***(3) below as for the existence of* α_l*) and put* $H := \sum_{s=1}^{m} h_s + \sum_{l=1,j=1}^{k\ \ \alpha_l} f_j^{(l)}$, *then*

(d) HRH *is a hereditary ring.*

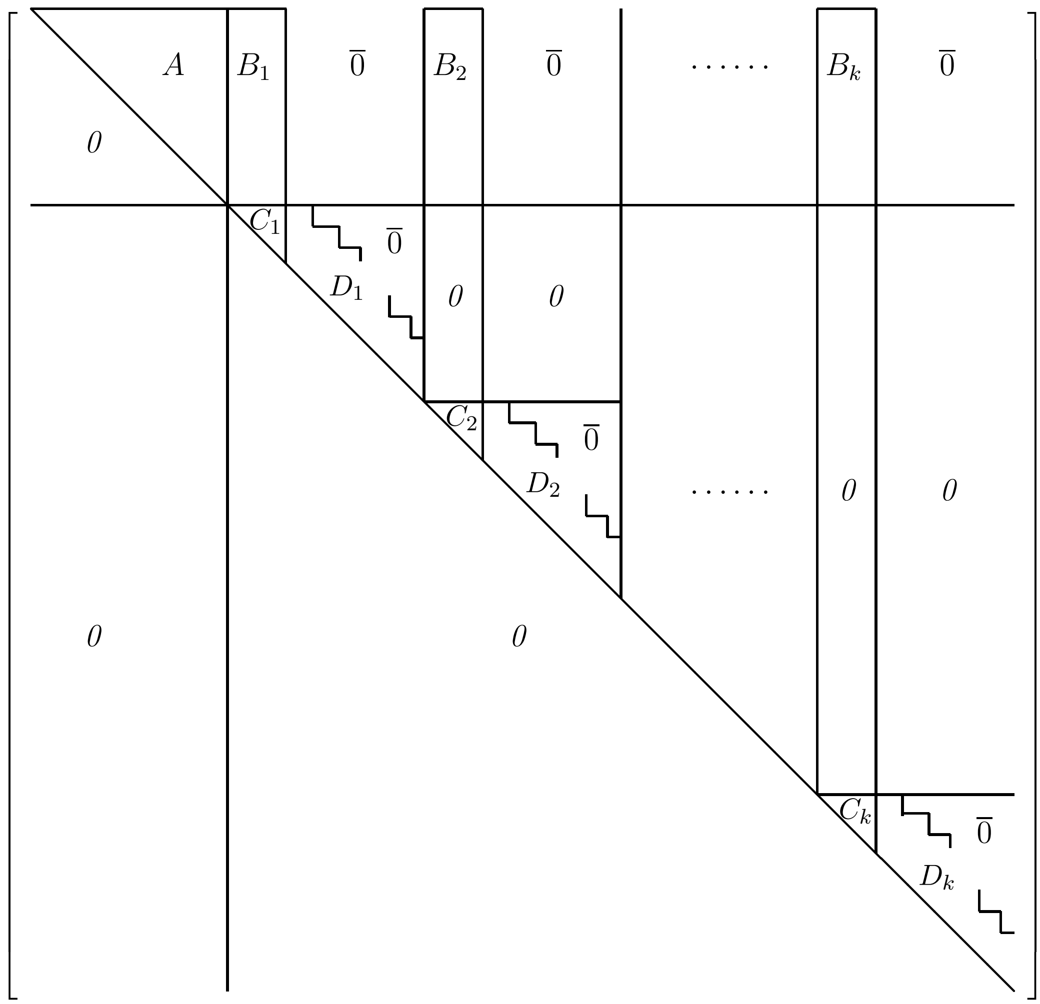
Remark 3 . *By [5] we know that a hereditary ring is represented as*

$$\begin{bmatrix} D_1 & M_{1,2} & M_{1,3} & \cdots & \cdots & M_{1,n} \\ 0 & D_2 & M_{2,3} & \cdots & \cdots & M_{2,n} \\ \vdots & & 0 & \cdots & \cdots & \\ & & & \ddots & \cdots & \\ & & & 0 & D_{n-1} & M_{n-1,n} \\ 0 & & \cdots & & 0 & D_n \end{bmatrix},$$

where $D_1, D_2, \cdots, D_n$ *are division rings and* M_{ij} *is a left* D_i*-right* D_j*-bimodule for any* i, j. *Further by [12] a serial ring in the first category is represented* *as the following factor ring:*

$$\left[\begin{array}{cccccccccccccccc}
D & D & \cdots & \cdots & \cdots & D & \overline{0} & & & & \cdots & & & & & \overline{0} \\
\ddots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots & & & & \cdots & & & & & \overline{0} \\
 & 0 & D & D & \cdots & D & \overline{0} & & & & \cdots & & & & & \overline{0} \\
 & & 0 & D & D & \cdots & & D & \overline{0} & & & \cdots & & & & \overline{0} \\
 & & & 0 & D & \cdots & & \vdots & \vdots & & & \cdots & & & & \overline{0} \\
\vdots & & & & 0 & \ddots & & D & \overline{0} & \cdots & & & & & & \overline{0} \\
\vdots & & & & & \ddots & & \ddots & & \cdots & \cdots & & D & \overline{0} & \cdots & \overline{0} \\
\vdots & & & & & & & \ddots & & \ddots & \cdots & & \vdots & \vdots & \cdots & \overline{0} \\
\vdots & & & & & & & & & 0 & D & \cdots & D & \overline{0} & \cdots & \overline{0} \\
 & & & & & & & & & & 0 & D & \cdots & & \cdots & D \\
 & & & & & & & & & & & 0 & D & & & \vdots \\
 & & & & & & & & & & & & 0 & & \ddots & \vdots \\
0 & & & & \cdots & \cdots & & \cdots & & \cdots & & & & & 0 & D
\end{array}\right],$$

where D is a division ring. So a ring R in Theorem 2(iii) is represented as the following factor ring:



,

where $1_A = \sum_{l=1}^{m} h_l$, $1_{C_l} = \sum_{j=1}^{\alpha_l} f_j^{(l)}$ *and* $1_{C_l+D_l} = \sum_{j=1}^{n_l} f_j^{(l)}$ *for each* l*. Further* $HRH = A \cup (\cup_{l=1}^{k}(B_l \cup C_l))$ *and* $S_lRS_l = C_l \cup D_l$*.*

In Theorem 2 a right almost hereditary ring is characterized by right ideals. Here we shall characterize a ring in Theorem 2(iii) by left ideals.

First we characterize α_l in Theorem 2(iii) not using the right module structure.

Lemma 4. *Let R be a ring satisfying $(a),(b)$ in Theorem 2(iii) and α_l as in Theorem 2(iii). Define an integer α'_l to satisfy $(h_1+\cdots+h_m)Rf_j^{(l)}=0$ for any $j=\alpha'_l+1,\cdots,n_l$ but $(h_1+\cdots+h_m)Rf_{\alpha'_l}^{(l)}\neq 0$. Then $\alpha_l=\alpha'_l$.*

Using Lemma 4 we have a lemma.

Lemma 5.

(1) Let R be a ring in Theorem 2(iii). We may assume that $h_sRh_t=0$ for any $s>t$ by the representation form of a hereditary ring (see Remark 3). Then the following condition (e) holds:

(e) $h_sJ \cong (\oplus_{i=s+1}^m (h_iR)^{u_i}) \oplus (\oplus_{l=1}^k (f_1^{(l)}R/\ f_1^{(l)}J^{\alpha_l})^{v_l})$ as right R-modules for some non-negative integers $u_{s+1},\cdots,u_m,v_1,\cdots,v_k$.

(2) Suppose that a ring R satisfies (a), (b), (e), then (c) and (d) hold.

Hence (a), (b), (c), (d) in Theorem 2(iii) can be replaced by (a), (b), (e).

The following gives a characterization of a ring in Theorem 2(iii) using left ideals.

Theorem 6. *Let R be a ring with $P(R)=\{h_1,\cdots,h_m,f_1^{(1)},f_2^{(1)},\cdots,f_{n_1}^{(1)},f_1^{(2)},\cdots,f_{n_2}^{(2)},f_1^{(3)},\cdots,f_{n_k}^{(k)}\}$. $P(R)$ satisfies (a),(b),(c),(d) in Theorem 2(iii) if and only if the following five condition hold for any $l=1,\cdots,k$, we put $S_l:=\sum_{j=1}^{n_l} f_j^{(l)}$,*

(a') S_lRS_l is a serial ring in the first category with $\{S_lRf_{n_l}^{(l)},S_lRf_{n_l-1}^{(l)},\cdots,S_lRf_1^{(l)}\}$ a Kupisch series of left S_lRS_l-modules,

(b') $S_lR(1-S_l)=0$ and $(h_1+\cdots+h_m)RS_l\neq 0$,

(c') $Jf_j^{(l)}/J^2 f_j^{(l)}$ *is simple as a left R-module for any $j = 2, \cdots, n_l$,*

we let α'_l be the same integer as in Lemma 4 and put $H' := \sum_{s=1}^{m} h_s + \sum_{l=1,j=1}^{k,\ \alpha'_l} f_j^{(l)}$, then

(d') $H'RH'$ *is a hereditary ring , and*

(f) $E({}_R Rf_1^{(l)}/Jf_1^{(l)})$ *is projective as a left R-module for any $l = 1, \cdots, k$.*

Then we note that $\alpha'_l = \alpha_l$, and so $H' = H$ and (d') coincides with (d), where H and (d) are as in Theorem 2(iii).

By using Theorem 6, we can show the following theorem, which is the main theorem in this paper.

Theorem 7. *R satisfies $(\#)_l$ if and only if R is a right almost hereditary ring.*

Remark 8 . *In [9, p801] M. Harada already showed that a right almost hereditary ring is not always a left almost hereditary ring. We shall give an example for this suggestion below. (see Example 11)*

§3 Strongly almost hereditary rings

Before considering right SAH rings, we define a special (serial) ring. A serial ring is called a *strongly serial ring* if it is a direct sum of indecomposable serial rings R with a Kupisch series $\{f_{1,1}R,\ f_{1,2}R, \cdots, f_{1,\beta_1}R, f_{2,1}R, \cdots, f_{m,\beta_m}R\}$ such that $|f_{i,\beta_i}R| = 2$ for any $i = 1, \cdots, m-1$ and $|f_{m,\beta_m}R| = 1$ or 2, where $P(R) = \{f_{i,j}\}_{i=1,j=1}^{m\ \ \beta_i}$ and $f_{i,j}R$ is injective iff $j = 1$. Then, if $|f_{m,\beta_m}R| = 1$ (resp. $= 2$), then R is a serial ring in the first (resp.

second) category. Further we can easily check the following characterization of a strongly serial ring.

The following is a structure theorem of a right SAH ring given by M. Harada.

Theorem 9 *([9, Theorem 3]).* *A ring is right SAH if and only if it is a direct sum of the following rings:*

(i) *Hereditary rings;*

(ii) *strongly serial rings;*

(iii) *rings R with $P(R) = \{h_1, \cdots, h_m, f_1^{(1)}, f_2^{(1)}, \cdots, f_{n_1}^{(1)}, f_1^{(2)}, \cdots, f_{n_2}^{(2)}, f_1^{(3)}, \cdots, f_{n_k}^{(k)}\}$ such that, for each $l = 1, \cdots, k$ we put $S_l := \sum_{j=1}^{n_l} f_j^{(l)}$ and $H := \sum_{s=1}^{m} h_s + \sum_{l=1}^{k} f_1^{(l)}$, the following three conditions hold for any $l = 1, \cdots, k$:*

(x) *S_lRS_l is a strongly serial ring in the first category with a Kupisch series $\{f_1^{(l)}RS_l, f_2^{(l)}RS_l, \cdots, f_{n_l}^{(l)}RS_l\}$ of right S_lRS_l-modules,*

(y) *$S_lR(1-S_l) = 0$, $(h_1+\cdots+h_m)Rf_1^{(l)} \neq 0$ and $(h_1+\cdots+h_m)R(f_2^{(l)} + \cdots + f_{n_l}^{(l)}) = 0$, and*

(z) *HRH is a hereditary ring.*

We note that by Lemma 4 a ring in Theorem 9 (iii) coincides with a ring in Theorem 2 (iii) if it satisfies that $\alpha_l = 1$ and S_lRS_l is a strongly serial ring for any $l = 1, \cdots, k$, where α_l and S_l are as in it.

Moreover, the condition (ii) in the above Theorem is not the same as [9, Theorem 3], i.e., when R is a serial ring in the second category, he wrote that "R is a serial ring in the second category with $J^2 = 0$". But this original condition is not suitable. We give an example. Let R be a serial ring in the second category with $P(R) = \{f_1, f_2, f_3, f_4\}$ such that $\{f_1R, f_2R, f_3R, f_4R\}$ is a Kupisch series and $|f_1R| = 4,\ |f_2R| = 3,\ |f_3R| = 2,\ |f_4R| = 2$. Then R is a strongly serial ring. So it is right SAH by the following proof. But $J^2 \neq 0$. In an unpublished lecture note written by M. Harada the condition is already corrected.

The purpose of this section is to show the following theorem.

Theorem 10. *A ring R is right SAH if and only if R satisfies $(*^{\#})_l$.*

A right SAH ring does not always satisfy $(*^{\#})_r$ and a ring satisfying $(*^{\#})_r$ is not always a right SAH ring, Now we give an example.

Example 11. Consider a factor ring

$$R := \begin{bmatrix} D & D & 0 & D & \overline{0} & \overline{0} \\ 0 & D & 0 & D & \overline{0} & \overline{0} \\ 0 & 0 & D & D & \overline{0} & \overline{0} \\ 0 & 0 & 0 & D & D & \overline{0} \\ 0 & 0 & 0 & 0 & D & D \\ 0 & 0 & 0 & 0 & 0 & D \end{bmatrix},$$

where D is a division ring. And we consider that R is a ring by the ordinary addition and the multiplication of matrices. Put $H := e_1 + e_2 + e_3 + e_4$ and $S_1 := e_4 + e_5 + e_6$, where e_i is the (i, i)-matrix unit for any i.

Then HRH is a hereditary ring and S_1RS_1 is a strongly serial ring in the first category. And R is a ring in Theorem 9(iii), i.e., R is a right SAH ring.

But we claim that R does not satisfies $(*^{\#})_r$. e_4R is an injective left R-module with $e_4R/S(e_4R) \cong e_4R/e_4J$. And $e_4R/S(e_4R)$ is not injective.

Further $e_4R/S(e_4R)$ is not almost injective by [10, Corollary $1^\#$] since $e_1R \oplus e_3R$ is a projective cover of $E(e_4R/e_4J)$.

By Theorem 10 R satisfies $(*^\#)_l$ but is not a left SAH ring.

§4 Stronger conditions than that of a SAH ring

The following is a structure theorem of an artinian ring which satisfies $(**)_r$ and $(***)_r$ which are stronger conditions than that of a right SAH ring:

Theorem 12 *([9, Theorem 4]). For a ring the following are equivalent:*

(a) It satisfies $(**)_r$;

(b) it satisfies $(***)_r$;

(c) it is a direct sum of the following rings:

(i) Hereditary rings which are not serial;

(ii) serial rings with the radical square zero;

(iii) rings R *in Theorem 9 (iii) such that* HRH *is not a serial ring and* $J(S_lRS_l)^2 = 0$ *for any* $l = 1, \cdots, k$, *where* H *and* S_l *are as in Theorem 9 (iii).*

The purpose of this section is to show the following theorem.

Theorem 13. *For a ring* R *the following are equivalent:*

(a) R *satisfies* $(**)_r$ $(\Leftrightarrow (***)_r)$;

(b) R *satisfies* $(**^\#)_l$;

(c) R *satisfies* $(***^\#)_l$.

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