Stable equivalence functors and syzygy functors

Yosuke OHNUKI

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Tokyo University of Agriculture and Technology, 2-24-16 Nakacho, Koganei, Tokyo 184-8588, Japan
E-mail: ohnuki@cc.tuat.ac.jp

In this note, K is a fixed field. An algebra means a finite dimensional selfinjective K-algebra. An module means a finite dimensional left module. For an algebra Λ, we denote by modΛ the category of Λ-modules. The stable category of an algebra Λ is denoted by $\underline{\text{mod}}\Lambda$.

The triangularity of stable categories was stedied by Happel, Rickard and many mathematcian [2], [3], [5], [9], [10]. Our aim in this note is to show that for an equivalence functor $F : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda'$, F is a triangle functor if and only if F commutes with the syzygy functors i.e., $F\Omega \simeq \Omega' F$, where Ω and Ω', respectively, are the syzygy functors of $\underline{\text{mod}}\Lambda$ and $\underline{\text{mod}}\Lambda'$, respectively. Note that a stable equivalence functor commutes with the syzygy functors if and only if it commutes with the Nakayama functors [1], and therefore if Λ and Λ' are stable equivalence symmetric algebras, then $\underline{\text{mod}}\Lambda$ and $\underline{\text{mod}}\Lambda'$ are triangle equivalent.

The detail in this note is referred to [6].

1 A stable category

Let Λ be an algebra. The stable (module) category $\underline{\text{mod}}\Lambda$ is the factor category of modΛ by morphisms factoring through projective Λ-modules. Objects of $\underline{\text{mod}}\Lambda$ are Λ-modules. For Λ-modules X, Y, a morphism from X to Y in $\underline{\text{mod}}\Lambda$ is given by its residue class in $\text{Hom}_\Lambda(X, Y)/\mathcal{P}(X, Y)$, where $\mathcal{P}(X, Y)$ is the subset of $\text{Hom}_\Lambda(X, Y)$ consisting of morphisms which factor through projective Λ-modules. For each morphism $u : X \to Y$ in modΛ, the residue class of u is denoted by $\underline{u}$.

For a Λ-module X, we denote by $\mu_X : X \to I_X$ the injective hull in modΛ, and let $\Omega^{-1}X$ be the cokernel of μ_X. Then for a morphism $u : X \to Y$ in modΛ, $\Omega^{-1}u$ is given by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\mu_X} & I_X & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \\ & & {\scriptstyle u}\big\downarrow & & \big\downarrow & & {\scriptstyle \Omega^{-1}u}\big\downarrow & & \\ 0 & \longrightarrow & Y & \xrightarrow{\mu_Y} & I_Y & \longrightarrow & \Omega^{-1}Y & \longrightarrow & 0. \end{array}$$

Note that $\Omega^{-1}u$ is uniquely determined by u in $\underline{\text{mod}}\Lambda$ i.e., u factors through projective Λ-modules if and only if so does $\Omega^{-1}u$. Then $\Omega^{-1}u$ is denoted by $\Omega^{-1}\underline{u}$, and $\Omega^{-1} : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda$ is the equivalence functor [4]. The quasi-inverse of Ω^{-1} is called the syzygy (or the Heller's loop-space) functor.

For a morphism $u : X \to Y$ in modΛ, we have the following commutative diagram with exact rows in modΛ

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\mu_X} & I_X & \xrightarrow{\pi_{\Omega^{-1}X}} & \Omega^{-1}X & \longrightarrow & 0 \\ & & {\scriptstyle u}\big\downarrow & & {\scriptstyle x}\big\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{v} & C(u) & \xrightarrow{w} & \Omega^{-1}X & \longrightarrow & 0. \end{array} \tag{1}$$

$C(u)$ is called a mapping cone of u. Note that a mapping cone is uniquely determined by a morphism in $\underline{\text{mod}}\Lambda$, and $C(u)$ is also called a mapping cone of a morphism $\underline{u}$ in $\underline{\text{mod}}\Lambda$. The diagram (1) induces the sequence $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C(u) \xrightarrow{\underline{w}} \Omega^{-1}X$ which is called a standard triangle in $\underline{\text{mod}}\Lambda$. Let $\mathcal{T}$ be a collection of sextuples which are isomorphic to standard triangles in $\underline{\text{mod}}\Lambda$. The stable category $\underline{\text{mod}}\Lambda$ is regarded as a triangulated category [2], [3] whose translation is the equivalence functor $\Omega^{-1} : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda$, and $\mathcal{T}$ is the collection of triangles.

A short exact sequence $0 \to X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$ is called quasi-indecomposable provided that if there exist short exact sequences $0 \to Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{g_1} Y_3 \to 0$ and $0 \to Z_1 \xrightarrow{f_2} Z_2 \xrightarrow{g_2} Z_3 \to 0$ such that $X_i \simeq Y_i \oplus Z_i$ $(i = 1, 2, 3)$ and $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$, then either Y_i are projective for any i and Z_j is nonzero nonprojective for some j, or Z_i are projective for any i and Y_j is nonzero nonprojective for some j. Then, $0 \to X_1 \xrightarrow{v} X_2 \xrightarrow{w} X_3 \to 0$ is quasi-indecomposable if and only if the induced triangle $(X_1, X_2, X_3, \underline{v}, \underline{w}, \underline{u})$ is indecomposable in $\underline{\text{mod}}\Lambda$.

2 A stable equivalence functor which commutes with the syzygy functors

Let Λ and Λ' be algebras and $F : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda'$ an equivalence functor. We denote by Ω and Ω', respectively, are the syzygy functors of Λ and Λ', respectively. For a morphism $\underline{u}$ in $\underline{\mathrm{mod}}\Lambda$, the mapping cone of $\underline{u}$ denotes by $C(u)$, and for a morphism $\underline{u}'$ in $\underline{\mathrm{mod}}\Lambda'$, the mapping cone of $\underline{u}'$ denotes by $C'(u')$. We denote by $\iota_{X'} : X' \to Q_{X'}$ the injective hull of a Λ'-module X'.

Lemma 2.1 ([6]). *Assume that $F : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda'$ is an equivalence functor with $\Omega' F \simeq F\Omega$. For a morphism $u : X \to Y$ in $\mathrm{mod}\Lambda$, we set $\underline{u}' := F(\underline{u})$. Let*

$$0 \longrightarrow X \overset{\begin{pmatrix} u \\ \mu_X \end{pmatrix}}{\longrightarrow} Y \oplus I_X \overset{(v \ \ -x)}{\longrightarrow} C(u) \longrightarrow 0$$

be a quasi-indecomposable short exact sequence in $\mathrm{mod}\Lambda$, and

$$0 \longrightarrow FX \overset{\begin{pmatrix} u' \\ \iota_{FX} \end{pmatrix}}{\longrightarrow} FY \oplus Q_{FX} \overset{(v_1' \ \ -x_1')}{\longrightarrow} C'(u') \longrightarrow 0$$

a short exact sequence in $\mathrm{mod}\Lambda'$. Then there exists an isomorphism $\underline{z}$: $C(u) \to F^{-1}C'(u')$ in $\underline{\mathrm{mod}}\Lambda$ such that $F(\underline{zv}) = \underline{v_1}'$.

An additive functor $F : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda'$ is called a triangle (or exact) functor provided that $\Omega'^{-1}F \simeq F\Omega^{-1}$ and sextuple $(FX, FY, FZ, F\underline{u}, F\underline{v}, F\underline{w})$ is a triangle in $\underline{\mathrm{mod}}\Lambda'$ for any triangle $(X, Y, Z, \underline{u}, \underline{v}, \underline{w})$ in $\underline{\mathrm{mod}}\Lambda$. If F is an equivalence and a triangle functor, then $\underline{\mathrm{mod}}\Lambda$ and $\underline{\mathrm{mod}}\Lambda'$ are equivalent as triangulated categories.

Theorem 2.2 ([6]). *The following conditions are equivalent for an equivalence functor $F : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda'$*

1. *F is a triangle functor.*

2. *F commutes with the syzygy functors i.e., $\Omega' F \simeq F\Omega$.*

The outline of proof. Assume that $\Omega' F \simeq F\Omega$ in order to prove 2 $\Longrightarrow$ 1. Note that $\underline{\mathrm{mod}}\Lambda$ is a Krull-Schmidt category, because so is $\mathrm{mod}\Lambda$. Any triangle in $\underline{\mathrm{mod}}\Lambda$ decomposes finite direct sum of indecomposable triangles.

Let $\underline{u}$ be a morphism in $\underline{\mathrm{mod}}\Lambda$ with an indecomposable standard triangle $(X, Y, C(u), \underline{u}, \underline{v}, \underline{w})$ in $\underline{\mathrm{mod}}\Lambda$, and we use the notation of Lemma 2.1. In particular, let $(FX, FY, C'(u'), \underline{u}', \underline{v_1}', \underline{w_1}')$ be a standard triangle in $\underline{\mathrm{mod}}\Lambda'$.

We have the following commutative diagram with exact rows in modΛ

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{\mu_Y} & I_Y & \xrightarrow{\pi_{\Omega^{-1}Y}} & \Omega^{-1}Y & \longrightarrow & 0 \\
 & & {\scriptstyle v}\big\downarrow & & \begin{pmatrix} 0 \\ 1 \end{pmatrix}\big\downarrow & & \| & & \\
0 & \longrightarrow & C(u) & \xrightarrow{\begin{pmatrix} w \\ s \end{pmatrix}} & \Omega^{-1}X \oplus I_Y & \xrightarrow{\begin{pmatrix} -\Omega^{-1}u & \pi_{\Omega^{-1}Y} \end{pmatrix}} & \Omega^{-1}Y & \longrightarrow & 0.
\end{array}$$

Note that the bottom rows is quasi-indecomposable. We also have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & FX & \xrightarrow{\iota_{FX}} & Q_{FX} & \xrightarrow{\rho_{\Omega'^{-1}FX}} & \Omega'^{-1}FX & \longrightarrow & 0 \\
 & & {\scriptstyle u'}\big\downarrow & & {\scriptstyle x'_1}\big\downarrow & & \| & & \\
0 & \longrightarrow & FY & \xrightarrow{v'_1} & C'(u') & \xrightarrow{w'_1} & \Omega'^{-1}FX & \longrightarrow & 0 \\
 & & \| & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & FY & \xrightarrow{\begin{pmatrix} v' \\ \iota_{FY} \end{pmatrix}} & FC(u) \oplus Q_{FY} & \xrightarrow{t'} & \Omega'^{-1}FX \oplus Q & \longrightarrow & 0 \\
 & & \| & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & FY & \xrightarrow{v'_1} & C'(u') & \xrightarrow{w'_1} & \Omega'^{-1}FX & \longrightarrow & 0
\end{array}$$

where $\underline{t'} \simeq \underline{F\begin{pmatrix} w & 0 \\ s & -1 \end{pmatrix}} = F\underline{w}$ and Q is some projective Λ'-module, by Lemma 2.1. Since the top and bottom rows induce a standard triangle, $(FX, FY, FC(u), F\underline{u}, F\underline{v}, F\underline{w})$ is also a triangle in $\underline{\text{mod}}\Lambda'$. F is a triangle functor. $\square$

In [1, Chapter X], we obtain the following corollary.

Corollary 2.3. *Let $F : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda'$ be an equivalence functor. If it holds $\mathcal{N}'F \simeq F\mathcal{N}$, then F is a triangle functor. In particular, if Λ and Λ' are symmetric algebras, then any equivalence functor $F : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda'$ is a triangle functor.*

In [7], we constructed the trivial extension algebra and the non-symmetric Hochschild extension algebra given by 2-cocycle. They are stably equivalent by [8, Example 4.4]. Therefore, we obtain the non-triangle stable equivalence functor. Howevere, we do not know the example which is a non-triangle stable equivalence functor if base field is algebraically closed.

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