LEXICOGRAPHIC GRÖBNER BASES OF TORIC IDEALS ARISING FROM ROOT SYSTEMS

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Abstract. The present paper is a brief draft based on a joint work with Takayuki Hibi. Gröbner bases of toric ideals arising from root systems are studied.

Introduction

Let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set and let $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ denote the Laurent polynomial ring over a field K. We associate each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with the monomial $\mathbf{t}^\alpha s = t_1^{\alpha_1} \cdots t_n^{\alpha_n} s \in K[\mathbf{t}, \mathbf{t}^{-1}, s]$ and write $\mathcal{R}_K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by all monomials $\mathbf{t}^\alpha s$ with $\alpha \in \mathcal{A}$. Let $K[\mathbf{x}] = K[\{x_\alpha \,;\, \alpha \in \mathcal{A}\}]$ denote the polynomial ring in $\sharp(\mathcal{A})$ variables over K and $I_\mathcal{A} \subset K[\mathbf{x}]$ the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to \mathcal{R}_K[\mathcal{A}]$ defined by setting $\pi(x_\alpha) = \mathbf{t}^\alpha s$ for all $\alpha \in \mathcal{A}$. The ideal $I_\mathcal{A}$ is called the *toric ideal* of the configuration $\mathcal{A}$. It is known [9] that if $I_\mathcal{A}$ possesses a squarefree initial ideal, then the convex hull of $\mathcal{A}$ possesses a unimodular triangulation.

Fix $n \geq 2$. Let $\mathbf{e}_i$ denote the i-th unit coordinate vector of $\mathbb{R}^n$. We write $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$ for the set of positive roots of root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$, respectively ([3, pp. 64 – 65]):

$$\mathbf{A}_{n-1}^+ = \{\mathbf{e}_i - \mathbf{e}_j \,;\, 1 \leq i < j \leq n\};$$
$$\mathbf{B}_n^+ = \{\mathbf{e}_i \,;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \,;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j \,;\, 1 \leq i < j \leq n\};$$
$$\mathbf{C}_n^+ = \{2\mathbf{e}_i \,;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j \,;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j \,;\, 1 \leq i < j \leq n\};$$
$$\mathbf{D}_n^+ = \{\mathbf{e}_i + \mathbf{e}_j \,;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j \,;\, 1 \leq i < j \leq n\};$$
$$\mathbf{BC}_n^+ = \mathbf{B}_n^+ \cup \mathbf{C}_n^+.$$

Let, in addition, $\widetilde{\Phi}^+ = \Phi^+ \cup \{(0, 0, \ldots, 0)\}$, where $\Phi = \mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{BC}_n$ and where $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$.

In their combinatorial study of hypergeometric functions associated with root systems, Gelfand, Graev and Postnikov [2, Theorem 6.3] discovered a squarefree quadratic initial ideal of the toric ideal $I_{\widetilde{\mathbf{A}}_{n-1}^+}$ of $\widetilde{\mathbf{A}}_{n-1}^+$. Moreover, for *any* subconfiguration $\mathcal{A}$ of $\mathbf{A}_{n-1}^+$, the configuration $\widetilde{\mathcal{A}} = \mathcal{A} \cup (0, 0, \ldots, 0)$ possesses a regular unimodular triangulation ([7, Example 2.4 (a)]). Stanley [8, Exercise 6.31 (b), p. 234] computed the Ehrhart polynomial of the convex polytope $\text{conv}(\widetilde{\mathbf{A}}_{n-1}^+)$. Fong [1] constructed certain triangulations of the configurations $\widetilde{\mathbf{B}}_n^+$ $(= \text{conv}(\widetilde{\mathbf{D}}_n^+) \cap \mathbb{Z}^n)$ and $\text{conv}(\widetilde{\mathbf{C}}_n^+) \cap \mathbb{Z}^n$ $(= \widetilde{\mathbf{BC}}_n^+)$, and computes the Ehrhart polynomials of $\text{conv}(\widetilde{\mathbf{B}}_n^+)$

and $\mathrm{conv}(\widetilde{\mathbf{C}}_n^+)$. The triangulations studied in [1] are, however, non-unimodular. Motivated by their results, Ohsugi–Hibi [6] showed that

Proposition 0.1. *Let $\Phi \subset \mathbb{Z}^n$ be one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. Then, there exists a reverse lexicographic order such that the initial ideal of $I_{\widetilde{\Phi}^+}$ is generated by squarefree quadratic monomials.*

Moreover, Ohsugi–Hibi [5] discussed subconfigurations $\widetilde{\mathcal{A}} = \mathcal{A} \cup \{(0,0,\ldots,0)\}$ of $\widetilde{\mathbf{B}}_n^+ \cup \widetilde{\mathbf{C}}_n^+$ which possesses a (regular) unimodular triangulation (i.e., $I_{\widetilde{\mathcal{A}}}$ which possesses a squarefree initial ideal).

Hence, it is natural to study the same problem as above for I_{Φ^+} where $\Phi \subset \mathbb{Z}^n$ is one of the root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. (Then, I_{Φ^+} is not generated by quadratic binomials if $n \geq 6$.)

1. Squarefree lexicographic initial ideals

Let $\Phi^+ \subset \mathbb{Z}^n$ denote one of the configurations $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$. Let $K[\mathbf{A}_{n-1}^+]$, $K[\mathbf{B}_n^+]$, $K[\mathbf{C}_n^+]$, $K[\mathbf{D}_n^+]$ and $K[\mathbf{BC}_n^+]$ denote the polynomial rings

$$\begin{aligned}
K[\mathbf{A}_{n-1}^+] &= K[\{f_{i,j}\}_{1\leq i<j\leq n}],\\
K[\mathbf{B}_n^+] &= K[\{y_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}],\\
K[\mathbf{C}_n^+] &= K[\{a_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}],\\
K[\mathbf{D}_n^+] &= K[\{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}],\\
K[\mathbf{BC}_n^+] &= K[\{a_i\}_{1\leq i\leq n} \cup \{y_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}]
\end{aligned}$$

over K. Write $\pi : K[\Phi^+] \to K[\mathbf{t}, \mathbf{t}^{-1}, s]$ for the homomorphism defined by setting

$$\pi(a_i) = t_i^2 s, \quad \pi(y_i) = t_i s, \quad \pi(e_{i,j}) = t_i t_j s, \quad \pi(f_{i,j}) = t_i t_j^{-1} s.$$

Thus the kernel of π is the toric ideal I_{Φ^+}.

First, an explicit initial ideals of $I_{\mathbf{A}_{n-1}^+}$ generated by squarefree monomials of degree ≤ 3 will be constructed. Let $<_{lex}$ be the lexicographic order induced by the ordering of variables

$$f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n},$$

and let $<_{rev}$ be the reverse lexicographic order induced by the ordering of variables

$$f_{n-1,n} > f_{n-2,n} > f_{n-2,n-1} > \cdots > f_{2,3} > f_{1,n} > \cdots > f_{1,3} > f_{1,2}.$$

Then, the reduced Gröbner basis with respect to $<_{lex}$ (and $<_{rev}$) is as follows.

Theorem 1.1 ([4]). *The set of the binomials*

$$\begin{array}{cc}
f_{i,\ell}f_{j,k} - f_{i,k}f_{j,\ell}, & i<j<k<\ell,\\
f_{i,j}f_{j,k} - f_{i,i+1}f_{i+1,k}, & i+1<j<k,\\
f_{i,j}f_{k,k+1}f_{k+1,\ell} - f_{i,i+1}f_{i+1,j}f_{k,\ell}, & i+1<j<k<\ell-1,
\end{array}$$

is the reduced Gröbner basis of the toric ideal $I_{\mathbf{A}_{n-1}^+}$ with respect to both $<_{lex}$ and $<_{rev}$, where the initial monomial of each binomial is the first monomial.

Then, we can associate the initial ideal of $I_{\mathbf{A}_{n-1}^+}$ with respect to $<_{lex}$ with the regular unimodular triangulation $\Delta_{<_{lex}}$. A graph-theoretical characterization of the maximal faces of the triangulation $\Delta_{<_{lex}}$ is given in [4].

Second, we discuss the existence of squarefree initial ideals of the toric ideal I_{Φ^+} where $\Phi \subset \mathbb{Z}^n$ is one of the root systems $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. The similar argument as in [5] plays an important role in the proof of Theorems 1.2 and 1.4.

Let $<_{lex}^c$ be the lexicographic order induced by the ordering of variables

$$\begin{aligned}
&a_1 > a_2 > \cdots > a_n \\
&> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n} \\
&> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n}.
\end{aligned}$$

Theorem 1.2. *The initial ideal of the toric ideal $I_{\mathbf{C}_n^+}$ with respect to $<_{lex}^c$ is generated by squarefree monomials.*

Let $<_{lex}^d$ denote the lexicographic order obtained by restricting $<_{lex}^c$ to $K[\mathbf{D}_n^+]$. By the elimination property of the lexicographic order $<_{lex}^c$, we have the following corollary from Theorem 1.2.

Corollary 1.3. *The initial ideal of the toric ideal $I_{\mathbf{D}_n^+}$ with respect to $<_{lex}^d$ is generated by squarefree monomials.*

We now consider the root systems $\mathbf{B}_n$ and $\mathbf{BC}_n$. Let $<_{lex}^{bc}$ be the lexicographic order induced by the ordering of variables

$$\begin{aligned}
&a_1 > a_2 > \cdots > a_n \\
&> e_{n-1,n} > e_{n-2,n-1} > e_{n-2,n-1} > \cdots > e_{1,2} > e_{1,3} > \cdots > e_{1,n} \\
&> y_1 > y_2 > \cdots > y_n \\
&> f_{n-1,n} > f_{n-2,n-1} > f_{n-2,n-1} > \cdots > f_{1,2} > f_{1,3} > \cdots > f_{1,n}.
\end{aligned}$$

Theorem 1.4. *The initial ideal of the toric ideal $I_{\mathbf{BC}_n^+}$ with respect to $<_{lex}^{bc}$ is generated by squarefree monomials.*

Let $<_{lex}^b$ denote the lexicographic order obtained by restricting $<_{lex}^{bc}$ to $K[\mathbf{B}_n^+]$. By the elimination property of the lexicographic order $<_{lex}^{bc}$, we have the following corollary from Theorem 1.4.

Corollary 1.5. *The initial ideal of the toric ideal $I_{\mathbf{B}_n^+}$ with respect to $<_{lex}^b$ is generated by squarefree monomials.*

Remark 1.6. Let $n \geq 6$ and let Φ^+ denote one of the configurations $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$. Then I_{Φ^+} is not generated by quadratic binomials. Hence, in particular, I_{Φ^+} does not possess a quadratic Gröbner basis.

References

[1] W. Fong, Triangulations and Combinatorial Properties of Convex Polytopes, Dissertation, M. I. T., June, 2000.

[2] I. M. Gelfand, M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, *in* "Arnold–Gelfand Mathematics Seminars, Geometry and Singularity Theory" (V. I. Arnold, I. M. Gelfand, M. Smirnov and V. S. Retakh, Eds.), Birkhäuser, Boston, 1997, pp. 205 – 221.

[3] J. E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Second Printing, Revised, Springer–Verlag, Berlin, Heidelberg, New York, 1972.

[4] T. Kitamura, H. Ohsugi and T. Hibi, Gröbner bases associated with positive roots and Catalan numbers, preprint.

[5] H. Ohsugi and T. Hibi, Unimodular triangulations and coverings of configurations arising from root systems, *J. Algebraic Combinatorics*, **14** (2001), 199 – 219.

[6] H. Ohsugi and T. Hibi, Quadratic initial ideals of root systems, *Proc. Amer. Math. Soc.*, **130** (2002), 1913 – 1922.

[7] R. P. Stanley, Decompositions of rational convex polytopes, *Ann. Discrete Math.* **6** (1980), 333 – 342.

[8] R. P. Stanley, "Enumerative Combinatorics, Volume II," Cambridge University Press, Cambridge, New York, Sydney, 1999.

[9] B. Sturmfels, "Gröbner Bases and Convex Polytopes," Amer. Math. Soc., Providence, RI, 1995.

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