COHEN-MACAULAY MODULES AND HOLONOMIC MODULES OVER FILTERED RINGS

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ABSTRACT. We study Gorenstein dimension and grade of a module M over a filtered ring whose assosiated graded ring is a commutative Noetherian ring. An equality or an inequality between these invariants of a filtered module and its associated graded module is the most valuable property for an investigation of filtered rings. We prove an inequality $\text{G-dim}M \leq \text{G-dim}\,\text{gr}M$ and an equality $\text{grade}M = \text{grade}\,\text{gr}M$, whenever Gorenstein dimension of $\text{gr}M$ is finite (Theorems 2.3 and 2.8). We would say that the use of G-dimension adds a new viewpoint for studying filtered rings and modules. We apply these results to a filtered ring with a Cohen-Macaulay or Gorenstein associated graded ring and study a Cohen-Macaulay, perfect or holonomic module.

1. INTRODUCTION

Homological theory of filtered (non-commutative) rings grew in studying , among others, D-modules, i.e., rings of differential operators (cf. [4], [17] etc.). The use of an invariant 'grade' is a core of the theory for Auslander regular or Gorenstein filtered rings ([4], [5], [6], [7], [14]). In particular, its invariance under forming associated graded modules is essential. Using Gorenstein dimension ([1], [9]), we extend the class of rings for which the invariance holds.

Let Λ be a left and right Noetherian ring. Let $\text{mod}\Lambda$ (respectively, $\text{mod}\Lambda^{\text{op}}$) be the category of all finitely generated left (respectively, right) Λ-modules. We denote the stable category by $\underline{\text{mod}}\Lambda$, the syzygy functor by $\Omega : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda$, and the transpose functor by $\text{Tr} : \underline{\text{mod}}\Lambda \to \underline{\text{mod}}\Lambda^{\text{op}}$ (see [2], Chapter 4, §1 or [1], Chapter 2, §1). For $M \in \text{mod}\Lambda$, we put $M^* := \text{Hom}_\Lambda(M, \Lambda) \in \text{mod}\Lambda^{\text{op}}$.

Gorenstein dimension, one of the most valuable invariants of the homological study of rings and modules, is introduced in [1]. A Λ-module M is said to have *Gorenstein dimension zero*, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\text{Ext}^k_\Lambda(M, \Lambda) = \text{Ext}^k_{\Lambda^{\text{op}}}(M^*, \Lambda) = 0$ for $k > 0$. It follows from [1], Proposition 3.8 that G-dim $M = 0$ if and only if $\text{Ext}^k_\Lambda(M, \Lambda) = \text{Ext}^k_{\Lambda^{\text{op}}}(\text{Tr}M, \Lambda) = 0$ for $k > 0$. For a positive integer k, M is said to have *Gorenstein dimension less than or equal to* k, denoted by G-dim $M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with G-dim $G_i = 0$ for $0 \leq i \leq k$. We have that G-dim $M \leq k$ if and only if G-dim $\Omega^k M = 0$ by [1], Theorem 3.13. It is also proved in [1] that if G-dim $M < \infty$ then G-dim $M = \sup\{k : \text{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$. In the following, we abbreviate 'Gorenstein dimension' to G-dimension.

We define another important invariant 'grade'. Let $M \in \text{mod}\Lambda$. We put $\text{grade}_\Lambda M := \inf\{k : \text{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$.

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In this paper we study G-dimension and grade of a filtered module over a filtered ring whose assiciated graded ring is commutative and Noetherian and apply the results to a filtered ring with a Gorenstein or Cohen-Macaulay associated graded ring.

In section two, we study G-dimension and grade of modules over a filtered ring. As usual, we analyze them by using the properties of assiciated graded modules. We start from studying G-dimension. When an associated graded ring $\mathrm{gr}\Lambda$ of a filtered ring Λ is commutative and Noetherian, a filtered Λ-module M whose associated graded module $\mathrm{gr}M$ has finite G-dimension has also finite G-dimension and an inequality $\text{G-dim}M \leq \text{G-dim}\,\mathrm{gr}M$ holds true (Theorem 2.3). We see that if an associated graded ring is regular then an equality holds for every M. However, it is open whether an equality holds or not in general. As for G-dimension zero, we show that if $\text{G-dim}\,\mathrm{gr}M = 0$, then $\text{G-dim}M = 0$ and the converse holds whenever some additional conditions for M are assumed (Theorem 2.5). Assume further that $\mathrm{gr}\Lambda$ is a *local ring with the condition (P) (see Appendix), then 'Auslander-Bridger formula' holds for a filtered module M such that $\mathrm{gr}M$ has finite G-dimension and $\text{G-dim}M = \text{G-dim}\,\mathrm{gr}M$: $\text{G-dim}M + {}^*\mathrm{depth}\,\mathrm{gr}M = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda$ (Proposition 2.6).

To handle grade in the literatures, a kind of 'finitary' condition over a ring such as 'regularity' or 'Gorensteiness' is setted ([7], §5 and [14], Chapter Ⅲ, §2, 2.5). We find out that only the finiteness of G-dimension of $\mathrm{gr}M$ implies $\mathrm{grade}M = \mathrm{grade}\,\mathrm{gr}M$ for a filtered module with a good filtration (Theorem 2.8). Suppose that $\mathrm{gr}\Lambda$ is Gorenstein. Then all finite $\mathrm{gr}\Lambda$-modules have finite G-dimension. Thus all filtered modules with a good filtration satisfy the equality. Since regularity implies Gorensteiness, our results also cover regular filtered rings.

In section three, we apply the results obtained in the previous section to Cohen-Macaulay modules over filtered rings with a Cohen-Macaulay associated graded ring and holonomic modules over Gorenstein filtered rings. When $\mathrm{gr}\Lambda$ is a Cohen-Macaulay *local ring with the condition (P), we define Cohen-Macaulay filtered modules and see that they are perfect. Then they satisfy a duality (Theorem 3.2). Moreover, assume that Λ is Gorenstein. Then injective dimension of Λ is finite, say d, so that we can define a holonomic module. A filtered module M with a good filtration is holonomic, if $\mathrm{grade}M = d$. We generalize some results in [14], Chapter Ⅲ, §4 and give a characterization of a holonomic module M by a property of $\mathrm{Min}(\mathrm{gr}M)$. An example of a filtered (non-regular) Gorenstein ring is given in 3.8.

The summary of commutative graded Noetherian rings, especially, *local rings are stated in Appendix.

2. Gorenstein dimension and grade for modules over filtered Noetherian rings

Let Λ be a ring. A family $\mathcal{F} = \{\mathcal{F}_p\Lambda : p \in \mathbb{N}\}$ of additive subgroups of Λ is called a *filtration* of Λ, if

(i) $1 \in \mathcal{F}_0\Lambda$,

(ii) $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$,

(iii) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$,

(iv) $\Lambda = \cup_{p \in \mathbb{N}} \mathcal{F}_p \Lambda$.

A pair $(\Lambda, \mathcal{F})$ is called a *filtered ring*. In the following, a ring Λ is always a filtered ring for some filtration $\mathcal{F}$, so that we only say that Λ is a filtered ring.

Let $\sigma_p : \mathcal{F}_p\Lambda \to \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ be a natural homomorphism. Put

$$\mathrm{gr}\Lambda = \mathrm{gr}_{\mathcal{F}}\Lambda := \bigoplus_{p=0}^{\infty} \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda \quad (\mathcal{F}_{-1}\Lambda = 0).$$

Then $\mathrm{gr}\Lambda$ is a graded ring with multiplication

$$\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab), \quad a \in \mathcal{F}_p\Lambda,\ b \in \mathcal{F}_q\Lambda.$$

We always assume that $\mathrm{gr}\Lambda$ is a commutative Noetherian ring. Therefore, Λ is a right and left Noetherian ring. Our main objective is to study Λ by relating G-dimension and grade of $\mathrm{mod}\Lambda$ and those of $\mathrm{mod}(\mathrm{gr}\Lambda)$. Sometimes we assume further that $\mathrm{gr}\Lambda$ is a *local ring with the condition (P) (see Appendix).

Let M be a (left) Λ-module. A family $\mathcal{F} = \{\mathcal{F}_p M : p \in \mathbb{Z}\}$ of additive subgroups of M is called a *filtration* of M, if

(i) $\mathcal{F}_p M \subset \mathcal{F}_{p+1} M$,

(ii) $\mathcal{F}_{-p} M = 0$ for $p >> 0$,

(iii) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q M) \subset \mathcal{F}_{p+q} M$,

(iv) $M = \cup_{p \in \mathbb{Z}} \mathcal{F}_p M$.

A pair $(M, \mathcal{F})$ is called a *filtered* Λ-module. Similar to Λ, we sometimes abbreviate and say that M is a filtered module. Let $\tau_p : \mathcal{F}_p M \to \mathcal{F}_p M/\mathcal{F}_{p-1} M$ be a natural homomorphism. Put

$$\mathrm{gr}M = \mathrm{gr}_{\mathcal{F}}M := \bigoplus_{p \in \mathbb{Z}} \mathcal{F}_p M/\mathcal{F}_{p-1} M.$$

Then $\mathrm{gr}M$ is a graded $\mathrm{gr}\Lambda$-module by

$$\sigma_p(a)\tau_q(x) = \tau_{p+q}(ax), \quad a \in \mathcal{F}_p\Lambda,\ x \in \mathcal{F}_q M.$$

As for filtered rings and module, the reader is referred to [14] or [20]. We only state here some definitions and facts. For a filtered module $(M, \mathcal{F})$, we call $\mathcal{F}$ to be a *good filtration*, if there exist $p_k \in \mathbb{Z}$ and $m_k \in M$ $(1 \le k \le r)$ such that

$$\mathcal{F}_p M = \sum_{k=1}^{r} (\mathcal{F}_{p-p_k}\Lambda) m_k$$

for all $p \in \mathbb{Z}$. Then the following three conditions are equivalent ([14], Chapter I, 5.2 and [20], Chapter D, IV.3)

(a) M has a good filtration.

(b) $\mathrm{gr}_{\mathcal{F}}M$ is a finite $\mathrm{gr}\Lambda$-module for a filtration $\mathcal{F}$.

(c) M is a finitely generated Λ-module.

Therefore, we only consider a good filtration for a finitely generated Λ-module M, so that $\mathrm{gr}M$ is a finite $\mathrm{gr}\Lambda$-module.

Let $M,\ N$ be filtered Λ-modules. A Λ-homomorphism $f : M \to N$ is called a *filtered homomorphism*, if $f(\mathcal{F}_p M) \subset \mathcal{F}_p N$ for all $p \in \mathbb{Z}$. Further, f is called *strict*, if $f(\mathcal{F}_p M) =$

$\mathrm{Im}f \cap \mathcal{F}_pN$ for all $p \in \mathbb{Z}$. If M' is a submodule of M, then $\{M' \cap \mathcal{F}_pM : p \in \mathbb{Z}\}$, respectively $\{\mathcal{F}_pM + M'/M' : p \in \mathbb{Z}\}$ is a good filtration on M', respectively M/M'. We call them induced filtration on M' or M/M' and note that the canonical homomorphisms $M' \hookrightarrow M$ and $M \to M/M'$ are strict.

For a filtered homomorphism $f : M \to N$, we define a map $f_p : \mathcal{F}_pM/\mathcal{F}_{p-1}M \to \mathcal{F}_pN/\mathcal{F}_{p-1}N$ by $f_p(\tau_p(x)) = \tau_p(f(x))$ for $x \in \mathcal{F}_pM$. Then we define a $\mathrm{gr}\Lambda$-homomorphism

$$\mathrm{gr}f : \mathrm{gr}M = \oplus\mathcal{F}_pM/\mathcal{F}_{p-1}M \longrightarrow \mathrm{gr}N = \oplus\mathcal{F}_pN/\mathcal{F}_{p-1}N$$

by $\mathrm{gr}f := \oplus f_p$, so that $\mathrm{gr}f(\tau_p(x)) = \tau_p(f(x))$ for $x \in \mathcal{F}_pM$. It is easily seen that $\mathrm{gr}fg = (\mathrm{gr}f)(\mathrm{gr}g)$ for filtered homomorphisms $f : M \to N$ and $g : K \to M$.

For a filtered module M, an exact sequence

$$\cdots \longrightarrow F_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

is called a *filtered free resolution* of M, if all F_i are filtered free Λ-modules and all homomorphisms are strict filtered homomorphisms. We can always constract such a resolution with all F_i of finite rank for a finitely generated Λ-module (see [20], Chapter D, IV).

Let $M,\ N$ be filtered Λ-modules. We put, for $p \in \mathbb{Z}$,

$$\mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) = \{f \in \mathrm{Hom}_\Lambda(M,N) : f(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}N \text{ for all } q \in \mathbb{Z}\}$$

Then we have an ascending chain

$$\cdots \subset \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) \subset \mathcal{F}_{p+1}\mathrm{Hom}_\Lambda(M,N) \subset \cdots$$

of additive subgroups of $\mathrm{Hom}_\Lambda(M,N)$. Set

$$\mathrm{gr}\,\mathrm{Hom}_\Lambda(M,N) := \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)/\mathcal{F}_{p-1}\mathrm{Hom}_\Lambda(M,N)$$

Define an additive homomorphism

$$\varphi = \varphi(M,N) : \mathrm{gr}\,\mathrm{Hom}_\Lambda(M,N) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}N),\ \varphi(\tau_p(f))(\tau_q(x)) = \tau_{p+q}(f(x))$$

for $f \in \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N),\ x \in \mathcal{F}_qM$, where

$$\tau_p : \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) \longrightarrow \mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)/\mathcal{F}_{p-1}\mathrm{Hom}_\Lambda(M,N)$$

is a natural homomorphism for every $p \in \mathbb{Z}$. When M is a filtered module with a good filtration, the following facts hold (see [14], Chapter I, 6.9 or [20], Chapter D, VI.6):

(1) $\mathrm{Hom}_\Lambda(M,N) = \cup_{p\in\mathbb{Z}}\mathcal{F}_p\mathrm{Hom}_\Lambda(M,N)$.

(2) $\mathcal{F}_{-p}\mathrm{Hom}_\Lambda(M,N) = 0$ for $p >> 0$.

(3) φ is injective. Moreover, if M is a filtered free module, then it is bijective.

(4) When $N = \Lambda$, an additive group $\mathrm{Hom}_\Lambda(M,\Lambda)$ is a filtered Λ^{op}-module with a good filtration $\mathcal{F} := \{\mathcal{F}_p\mathrm{Hom}_\Lambda(M,N) : p \in \mathbb{Z}\}$ and φ is a $\mathrm{gr}\Lambda$-homomorphism.

Let $M \xrightarrow{f} N \xrightarrow{g} K$ be an exact sequence of filtered modules and filtered homomorphisms. Then $\mathrm{gr}M \xrightarrow{\mathrm{gr}f} \mathrm{gr}N \xrightarrow{\mathrm{gr}g} \mathrm{gr}K$ is exact (in $\mathrm{mod}\,\mathrm{gr}\Lambda$) if and only if f and g are strict (see [14], Chapter I, 4.2.4 or [20], Chapter D, III.3).

The following proposition is well-known.

2.1. PROPOSITION. *Let M be a filtered Λ-module with a good filtation. Then $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M,\Lambda)$ is a subfactor of $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ for $i \geq 0$.*

Proof. See [4], Chapter 2, 6.10 or [14], Chapter Ⅲ, 2.2.4. □

When G-dim gr$M = 0$, the functor Tr commutes with associated gradation.

2.2. Lemma. *Let M be a filtered Λ-module with a good filtation. Then there exists an epimorphism $\alpha : \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) \to \mathrm{gr}(\mathrm{Tr}_\Lambda M)$.*

Moreover, if $\text{G-dim}\,\mathrm{gr}M = 0$ *or* $\mathrm{grade}\,\mathrm{gr}M > 1$, *then α is an isomorphism.*

Proof. Take a filtered free resolution of M:

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0.$$

By definition, we have an exact sequence

$$F_0^* \xrightarrow{f_1^*} F_1^* \xrightarrow{g} \mathrm{Tr}_\Lambda M = \mathrm{Cok} f_1^* \longrightarrow 0,$$

where g is a canonical epimorphism. Let $\mathrm{Tr}_\Lambda M$ be equipped with the induced filtration by g. Then g is a strict filtered epimorphism. Let us consider the following diagrams in $\mathrm{mod}\,\mathrm{gr}\Lambda$ with the commutative squares and all the φ's isomorphisms:

$$(1)\quad \begin{array}{ccccccc} \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_0, \mathrm{gr}\Lambda) & \xrightarrow{(\mathrm{gr}f_1)^*} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_1, \mathrm{gr}\Lambda) & \longrightarrow & \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) & \longrightarrow & 0(\text{exact}) \\ \varphi\uparrow & & \varphi\uparrow & & & & \\ \mathrm{gr}F_0^* & \xrightarrow{\mathrm{gr}(f_1^*)} & \mathrm{gr}F_1^* & \xrightarrow{\mathrm{gr}g} & \mathrm{gr}(\mathrm{Tr}_\Lambda M) & \longrightarrow & 0 \end{array}$$

$$(2)\quad \begin{array}{ccccc} \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_0, \mathrm{gr}\Lambda) & \xrightarrow{(\mathrm{gr}f_1)^*} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_1, \mathrm{gr}\Lambda) & \xrightarrow{(\mathrm{gr}f_2)^*} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}F_2, \mathrm{gr}\Lambda) \\ \varphi\uparrow & & \varphi\uparrow & & \varphi\uparrow \\ \mathrm{gr}F_0^* & \xrightarrow{\mathrm{gr}(f_1^*)} & \mathrm{gr}F_1^* & \xrightarrow{\mathrm{gr}(f_2^*)} & \mathrm{gr}F_2^* \end{array}$$

Since the induced sequence $\cdots \to \mathrm{gr}F_1 \xrightarrow{\mathrm{gr}f_1} \mathrm{gr}F_0 \to \mathrm{gr}M \to 0$ is a free resolution of $\mathrm{gr}M$, the first row of (1) is exact. Since g is strict, $\mathrm{gr}g$ is surjective. Hence there exists a graded epimorphism $\alpha : \mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M) \to \mathrm{gr}(\mathrm{Tr}_\Lambda M)$. By assumption, we see that $\mathrm{Ext}^1_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = 0$, so that the first row of (2) is exact. There exists a filtered homomorphism $h : \mathrm{Tr}_\Lambda M \to F_2^*$ such that $f_2^* = h \circ g$. Since $\mathrm{gr}f_2^* = \mathrm{gr}h \circ \mathrm{gr}g$, we have $\mathrm{Im}\,\mathrm{gr}f_1^* \subset \mathrm{Ker}\,\mathrm{gr}g \subset \mathrm{Ker}\,\mathrm{gr}f_2^*$. The exactness of the second row of (2) implies $\mathrm{Im}\,\mathrm{gr}f_1^* = \mathrm{Ker}\,\mathrm{gr}f_2^*$. Thus $\mathrm{Im}\,\mathrm{gr}f_1^* = \mathrm{Ker}\,\mathrm{gr}g$, hence the second row of (1) is also exact, which implies that α is an isomorphism. □

2.3. Theorem. *Let M be a filtered Λ-module with a good filtration such that* $\mathrm{gr}M$ *is of finite G-dimension. Then G-dim$M \le$ G-dim gr$M$.*

Proof. We show that if $\text{G-dim}\,\mathrm{gr}M = k < \infty$, then $\text{G-dim}M \le k$. Let $k = 0$. Assume that $\text{G-dim}\,\mathrm{gr}M = 0$. For $i > 0$, since $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda)$ is a subfactor of $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$, we have $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$. Hence $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$. By Lemma 2.2, $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}\,\mathrm{Tr}_\Lambda M, \mathrm{gr}\Lambda) \cong \mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{Tr}_{\mathrm{gr}\Lambda}(\mathrm{gr}M), \mathrm{gr}\Lambda) = 0$ for $i > 0$. Hence $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}_\Lambda M, \Lambda) = 0$ as above. Thus $\text{G-dim}M = 0$.

Let $k > 0$. Since $\mathrm{gr}(\Omega^k M)$ and $\Omega^k(\mathrm{gr}M)$ are stably isomorphic(see [10], p.226 for the definition), the following holds:

$$\text{G-dim}\,\mathrm{gr}M \le k \Leftrightarrow \text{G-dim}\,\Omega^k(\mathrm{gr}M) = 0 \Leftrightarrow \text{G-dim}\,\mathrm{gr}(\Omega^k M) = 0.$$

Thus the statement holds by the case of $k = 0$. □

2.4. Corollary. *Assume that* $\mathrm{gr}\Lambda$ *is a* **local ring with the condition (P). If* $\mathrm{gr}\Lambda$ *is Gorenstein, then* $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda \leq {}^*\mathrm{depth}\,\mathrm{gr}\Lambda$.

Proof. Let M be a finitely generated Λ-module. Then M is a filtered module with a good filtration. Then $\text{G-dim}\,\mathrm{gr}M < \infty$ by Theorem A.9. Hence

$$\text{G-dim}M \leq \text{G-dim}\,\mathrm{gr}M = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda - {}^*\mathrm{depth}\,\mathrm{gr}M \leq {}^*\mathrm{depth}\,\mathrm{gr}\Lambda.$$

Therefore, $\mathrm{Ext}^i_\Lambda(M,\Lambda) = 0$ for all $i > {}^*\mathrm{depth}\,\mathrm{gr}\Lambda$, so that $\mathrm{id}_\Lambda\Lambda < \infty$. Similarly, we have $\mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda < \infty$. Thus $\mathrm{id}_\Lambda\Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda \leq {}^*\mathrm{depth}\,\mathrm{gr}\Lambda$. □

Thanks to Corollary 2.4, we call a filtered ring Λ a "*Gorenstein filtered ring*", if $\mathrm{gr}\Lambda$ is a Gorenstein *local ring with the condition (P).

We give a necessary and sufficient condition when $\text{G-dim}\,\mathrm{gr}M = 0$.

2.5. Theorem. *Let* M *be a filtered* Λ*-module with a good filtration. Then the following (1) and (2) are equivalent.*

(1) $\text{G-dim}\,\mathrm{gr}M = 0$.

(2) (2.1) $\text{G-dim}M = 0$.

(2.2) Suppose that $\cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$ *is a filtered free resolution of* M*, then all* f_i^* $(i > 0)$ *are strict.*

(2.2**) Suppose that* $\cdots \to G_1 \xrightarrow{g_1} G_0 \xrightarrow{g_0} M^* \to 0$ *is a filtered free resolution of* M^**, then all* g_i^* $(i > 0)$ *are strict.*

(2.3) A canonical map $\theta : M \to M^{**}$ *is strict.*

Moreover, under the above conditions, $\varphi_M : \mathrm{gr}M^* \to (\mathrm{gr}M)^*$ *and* $\varphi_{M^*} : \mathrm{gr}M^{**} \to (\mathrm{gr}M^*)^*$ *are isomorphisms, where* $\varphi_M = \varphi(M,\Lambda)$, $\varphi_{M^*} = \varphi(M^*,\Lambda)$.

Proof. (1) ⇒ (2): It follows from Theorem 2.3 that $\text{G-dim}M = 0$. From a filtered free resolution of M in (2.2), we get an exact sequence

$$0 \longrightarrow M^* \xrightarrow{f_0^*} F_0^* \xrightarrow{f_1^*} F_1^* \longrightarrow \cdots$$

This exact sequence and an exact sequence in $\mathrm{mod}\,\mathrm{gr}\Lambda$:

$$\cdots \longrightarrow \mathrm{gr}F_1 \longrightarrow \mathrm{gr}F_0 \longrightarrow \mathrm{gr}M \longrightarrow 0$$

induced from a resolution in (2.2) give the following commutative diagram

$$(*)\quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{gr}M^* & \xrightarrow{\mathrm{gr}(f_0^*)} & \mathrm{gr}F_0^* & \xrightarrow{\mathrm{gr}(f_1^*)} & \mathrm{gr}F_1^* & \longrightarrow & \cdots \\ & & \varphi\downarrow & & \varphi_0\downarrow & & \varphi_1\downarrow & & \\ 0 & \longrightarrow & (\mathrm{gr}M)^* & \longrightarrow & (\mathrm{gr}F_0)^* & \longrightarrow & (\mathrm{gr}F_1)^* & \longrightarrow & \cdots \end{array}$$

where $\varphi = \varphi(M,\Lambda)$, $\varphi_i = \varphi(F_i,\Lambda)$. Since $\text{G-dim}\,\mathrm{gr}M = 0$, the second row is exact. For $i \geq 0$, φ_i are isomorphisms. Thus a sequence

$$\mathrm{gr}F_0^* \xrightarrow{\mathrm{gr}(f_1^*)} \mathrm{gr}F_1^* \xrightarrow{\mathrm{gr}(f_2^*)} \mathrm{gr}F_2^* \longrightarrow \cdots$$

is exact, and so f_1^*, $f_2^*, \cdots$ are strict. Hence (2.2) holds. Since f_0 is a strict filtered epimorphism, f_0^* is a strict filtered monomorphism. Thus the first row of $(*)$ is exact. Therefore, $\varphi : \mathrm{gr}M^* \to (\mathrm{gr}M)^*$ is an isomorphism. Since $\text{G-dim}(\mathrm{gr}M)^* = 0$, we have $\text{G-dim}\,\mathrm{gr}M^* = 0$. Hence (2.2*) holds and φ_{M^*} is an isomorphism.

Let $\eta : \mathrm{gr}M \to (\mathrm{gr}M)^{**}$ be a canonical homomorphism. Consider the commutative diagram

$$(**)\qquad \begin{array}{ccc} \mathrm{gr}M & \xrightarrow{\mathrm{gr}\theta} & \mathrm{gr}M^{**} \\ \eta\downarrow & & \downarrow \varphi_{M^*} \\ (\mathrm{gr}M)^{**} & \xrightarrow{\varphi_M^*} & (\mathrm{gr}M^*)^*. \end{array}$$

Since η, φ_M^*, φ_{M^*} are isomorphisms, $\mathrm{gr}\theta$ is also an isomorphism. Thus θ is strict.

(2) $\Rightarrow$ (1): By (2.1) and (2.2), the first row of the diagram $(*)$ is exact. Thus the second row of $(*)$ is exact, so that $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = 0$ for $i > 0$ and $(\mathrm{gr}M)^* \cong \mathrm{gr}M^*$. Since G-dim$M^* = 0$, using the diagram $(*)$ obtained from (2.2*), we can show that $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M^*, \mathrm{gr}\Lambda) = 0$ for $i > 0$ and $(\mathrm{gr}M^*)^* \cong \mathrm{gr}M^{**}$. Thus we have $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}((\mathrm{gr}M)^*, \mathrm{gr}\Lambda) = 0$ for $i > 0$. By (2.3) and the above argument, the maps $\mathrm{gr}\theta$, φ_M^* and φ_{M^*} are isomorphisms in the diagram $(**)$, so that η is an isomorphism. Thus G-dim gr$M = 0$. □

Let filΛ be a category of all filtered Λ-modules with a good filtration and filtered homomorphisms. Let $\mathcal{G}$ be a subcategory of filΛ consisting of all filtered modules M whose associated graded module gr$M$ has finite G-dimension. It holds from Theorem 2.3 that a module in $\mathcal{G}$ has finite G-dimension. We further put a subcategory $\mathcal{G}_e$ of $\mathcal{G}$

$$\mathcal{G}_e := \{M \in \mathcal{G} : \text{G-dim}M = \text{G-dim}\,\mathrm{gr}M\}.$$

2.6. Proposition. *Assume that* grΛ *is a* **local ring with the condition (P). Let* $M \in \mathcal{G}_e$. *Then the following equality holds.*

$$\text{G-dim}M + {}^*\mathrm{depth}\,\mathrm{gr}M = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda.$$

Proof. The statement follows from Theorem A.8. □

2.7. Remarks. (i) It is interesting to know when $\mathcal{G}_e = \mathcal{G}$. If this is true, then we see that G-dim$M = 0$ if and only if G-dim gr$M = 0$ for $M \in \mathcal{G}$. Hence the condition (2.2), (2.2*), (2.3) in Theorem 2.5 are superfluous.

(ii) Suppose that pd$M$ is finite, where pd$M$ stands for a projective dimension of M. Then there is a filtered free resolution $0 \to F_k \to \cdots \to F_0 \to M \to 0$ of M. Since grF_i are free for $0 \le i \le k$, we see pd$M \ge$ pd gr$M$. It is well-known that if pd$M < \infty$ then pd$M =$ G-dim$M$ (e.g., [9], Proposition 1.2.10). Hence pd$M \le$ pd gr$M$ by 2.3, so that G-dim$M =$ G-dim gr$M$, whenever M has finite projective dimension. Especially, if Λ is of finite global dimension, then $\mathcal{G}_e = \mathcal{G}$.

(iii) Suppose that $0 \to M' \to M \to M'' \to 0$ is a strict exact sequence of filΛ. Then the followings are easy consequence of [9], Corollary 1.2.9 (b).

If $M', M'' \in \mathcal{G}_e$ and G-dim$M' >$ G-dimM'', then $M \in \mathcal{G}_e$.

If $M, M'' \in \mathcal{G}_e$ and G-dim$M >$ G-dimM'', then $M' \in \mathcal{G}_e$.

We shall study the another valuable invariant 'grade'. Its nicest feature that an equation $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ holds for a good filtered Λ-module M is proved when grΛ is regular (see e.g. [14]). We prove this equation under 'module-wise' conditions by which we can apply this equation fairly wide classes of filtered rings.

2.8. Theorem. *Let* Λ *be a filtered ring such that* grΛ *is a commutative Noetherian ring and* M *a filtered* Λ*-module with a good filtration. Assume that* gr$M$ *has finite G-dimension. Then an equality* $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ *holds.*

Proof. Put $s = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$. In order to show that $\mathrm{grade}_\Lambda M = s$, we must prove:

(i) $\mathrm{Ext}^s_\Lambda(M,\Lambda) \neq 0$,

(ii) $\mathrm{Ext}^i_\Lambda(M,\Lambda) = 0$ for $i < s$.

2.8.1. (cf. [14], Chapter III, §1) Let $\cdots \to F_i \xrightarrow{f_i} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \to 0$ be a filtered free resolution of M. Applying $(-)^*$ to it, we get a complex

$$F_\bullet : 0 \to F_0^* \xrightarrow{f_1^*} \cdots \to F_{i-2}^* \xrightarrow{f_{i-1}^*} F_{i-1}^* \xrightarrow{f_i^*} F_i^* \to \cdots$$

with each F_i^* filtered free and f_i^* a filtered homomorphism. We put, for $p, r, i \in \mathbb{N}$,

$$Z_p^r(i) := (f_i^*)^{-1}(\mathcal{F}_{p-r}F_i^*) \cap \mathcal{F}_pF_{i-1}^*, \quad Z_p^\infty(i) := \mathrm{Ker} f_i^* \cap \mathcal{F}_pF_{i-1}^*,$$

$$B_p^r(i) := f_{i-1}^*(\mathcal{F}_{p+r-1}F_{i-2}^*) \cap \mathcal{F}_pF_{i-1}^*, \quad B_p^\infty(i) := \mathrm{Im} f_{i-1}^* \cap \mathcal{F}_pF_{i-1}^*$$

Then the following sequence of inclusions holds:

$$Z_p^0(i) \supset Z_p^1(i) \supset \cdots \supset Z_p^\infty(i) \supset B_p^\infty(i) \supset \cdots \supset B_p^1(i) \supset B_p^0(i).$$

We put

$$E_p^r(i) := \frac{Z_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*}{B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*}, \quad E_i^r := \bigoplus_p E_p^r(i).$$

Then E_i^r is a $\mathrm{gr}\Lambda$-module for $r, i \geq 0$. When $r = 0$, we have

$$E_i^0 = \bigoplus_p \frac{(f_i^*)^{-1}(\mathcal{F}_pF_i^*) \cap \mathcal{F}_pF_{i-1}^* + \mathcal{F}_{p-1}F_{i-1}^*}{f_{i-1}^*(\mathcal{F}_{p-1}F_{i-2}^*) \cap \mathcal{F}_pF_{i-1}^* + \mathcal{F}_{p-1}F_{i-1}^*} = \bigoplus_p \frac{\mathcal{F}_pF_{i-1}^*}{\mathcal{F}_{p-1}F_{i-1}^*} = \mathrm{gr}F_{i-1}^*.$$

Hence we get a complex

$$E_\bullet^0 : 0 \to \mathrm{gr}F_0^* \to \cdots \to \mathrm{gr}F_i^* \to \cdots$$

which is an associated graded complex of $F_\bullet$. We show, for $r \geq 1$, that $\{E_i^r\}_{i\geq 0}$ also gives a complex $E_\bullet^r$. To do so, we define morphisms. By computation, it holds that

$$E_p^r(i) = \frac{Z_p^r(i)}{B_p^r(i) + Z_{p-1}^{r-1}(i)}, \quad f_i^*(Z_p^r(i)) = \mathcal{F}_{p-r}F_i^* \cap f_i^*(\mathcal{F}_pF_{i-1}^*) = B_{p-r}^{r+1}(i+1).$$

Thus the following hold:

(1) $f_i^*(Z_p^r(i)) = B_{p-r}^{r+1}(i+1) \subset Z_{p-r}^r(i+1)$,

(2) $f_i^*(B_p^r(i)) = 0$ and $f_i^*(Z_{p-1}^{r-1}(i)) = B_{p-r}^r(i+1)$.

We can show that f_i^* induces a map $\tilde{f}_p^r(i) : E_p^r(i) \to E_{p-r}^r(i)$, by

$$\tilde{f}_p^r(i)(x + B_p^r(i) + \mathcal{F}_{p-1}F_{i-1}^*) = f_i^*(x) + B_{p-r}^r(i+1) + Z_{p-r+1}^{r-1}(i+1) \ (x \in Z_p^r(i)).$$

Hence $\tilde{f}_p^r(i)(p \in \mathbb{N})$ give a graded $\mathrm{gr}\Lambda$-homomorphism

$$\tilde{f}_i^r : E_i^r = \bigoplus_p E_p^r(i) \longrightarrow E_{i+1}^r = \bigoplus_p E_p^r(i+1)$$

of degree $-r$. It is easily seen that $E_\bullet^r : \cdots \to E_i^r \xrightarrow{\tilde{f}_i^r} E_{i+1}^r \to \cdots$ is a complex.

2.8.2. Lemma. (cf. [14], p.130 (6)) *Under the above notation, we have $H^i(E_\bullet^r) \cong E_i^{r+1}$.*

Proof. We show

$$H(E_{p+r}^r(i-1) \xrightarrow{f} E_p^r(i) \xrightarrow{g} E_{p-r}^r(i+1)) \cong E_p^{r+1}(i),$$

where we put $f := \tilde{f}^r_{p+r}(i-1),\ g := \tilde{f}^r_p(i)$. Using (1) and (2), we can show that

$$x + B^r_p(i) + \mathcal{F}_{p-1}F^*_{i-1} \in \mathrm{Ker}g \Longleftrightarrow x \in (f^*_i)^{-1}(B^r_{p-r}(i+1) + \mathcal{F}_{p-r-1}F^*_i).$$

Thus we get

$$\mathrm{Ker}g = \frac{(Z^r_p(i) \cap (f^*_i)^{-1}(B^r_{p-r}(i+1) + \mathcal{F}_{p-r-1}F^*_i) + \mathcal{F}_{p-1}F^*_{i-1}}{B^r_p(i) + \mathcal{F}_{p-1}F^*_{i-1}}.$$

Further, we have

$$\mathrm{Im}f = \frac{f^*_{i-1}(Z^r_{p+r}(i-1)) + \mathcal{F}_{p-1}F^*_{i-1}}{B^r_p(i) + \mathcal{F}_{p-1}F^*_{i-1}}.$$

Hence the desired homology is

$$\begin{aligned}
\frac{\mathrm{Ker}g}{\mathrm{Im}f} &= \frac{(Z^r_p(i) \cap (f^*_i)^{-1}(B^r_{p-r}(i+1) + \mathcal{F}_{p-r-1}F^*_i) + \mathcal{F}_{p-1}F^*_{i-1}}{f^*_{i-1}(Z^r_{p+r}(i-1)) + \mathcal{F}_{p-1}F^*_{i-1}} \\
&= \frac{Z^{r-1}_{p-1}(i) + Z^r_p(i) \cap (f^*_i)^{-1}(\mathcal{F}_{p-r-1}F^*_i) + \mathcal{F}_{p-1}F^*_{i-1}}{f^*_{i-1}(Z^r_{p+r}(i-1)) + \mathcal{F}_{p-1}F^*_{i-1}} \\
&= \frac{Z^{r-1}_{p-1}(i) + Z^{r+1}_p + \mathcal{F}_{p-1}F^*_{i-1}}{f^*_{i-1}(Z^r_{p+r}(i-1)) + \mathcal{F}_{p-1}F^*_{i-1}} \\
&= \frac{Z^{r+1}_p(i) + \mathcal{F}_{p-1}F^*_{i-1}}{B^{r+1}_p(i) + \mathcal{F}_{p-1}F^*_{i-1}} \\
&= E^{r+1}_p(i),
\end{aligned}$$

where (2) (respectively, (1)) is used to show the second (respectively, fourth) equality. □

2.8.3. Corollary. *Assume that $E^1_{i-1} = 0$. Then we have $E^r_{i-1} = 0$ for $r \geq 1$ and there exists an exact sequence*

$$0 \to E^{r+1}_i \to E^r_i \to E^r_{i+1}$$

of $\mathrm{gr}\Lambda$*-modules for each $r \geq 1$.*

Proof. The first assertion directly follows from Lemma 2.8.2. Then the complex $E^r_\bullet$ yields an exact sequence $0 \to H^i(E^r_\bullet) \to E^r_i \to E^r_{i+1}$. Since $H^i(E^r_\bullet) \cong E^{r+1}_i$ by lemma 2.8.2, we get the desired exact sequence. □

2.8.4. We will show in this subsection that $E^r_{s+1} \neq 0$.

Condider the following commutative diagram

$$\begin{array}{ccccccccc}
E^0_\bullet = \mathrm{gr}F_\bullet: & 0 & \to & \mathrm{gr}F^*_0 & \to & \cdots & \to & \mathrm{gr}F^*_i & \to & \cdots \\
 & & & \wr\| & & & & \wr\| & & \\
 & 0 & \to & (\mathrm{gr}F_0)^* & \to & \cdots & \to & (\mathrm{gr}F_i)^* & \to & \cdots,
\end{array}$$

where rows are complexes and the second row is obtained by applying $\mathrm{Hom}_{\mathrm{gr}\Lambda}(-, \mathrm{gr}\Lambda)$ to a free resolution $\cdots \to \mathrm{gr}F_1 \to \mathrm{gr}F_0 \to \mathrm{gr}M \to 0$ of $\mathrm{gr}M$. Hence an isomorphism $E^1_{i+1} \cong \mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ holds by Lemma 2.8.2. (Note that $E^0_i \cong \mathrm{gr}F^*_{i-1}$.)

By assumption, we can apply A.15 to $\mathrm{gr}M$ and get the fact that grade $\mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = s$. Hence it holds that grade$E^1_{s+1} = s$ and $E^1_{i+1} = 0$ for $i < s$. By Corollary 2.8.3, we get an exact sequence of $\mathrm{gr}\Lambda$-modules

$$(3) \quad 0 \to E^{r+1}_{s+1} \to E^r_{s+1} \xrightarrow{\varphi} E^r_{s+2}.$$

By Lemma 2.8.2, E^r_{s+2} is a subfactor of E^{r-1}_{s+2} for $r \geq 1$. Thus every grΛ-submodule U of E^r_{s+2} is also a subfactor of $E^1_{s+2} = \mathrm{Ext}^{s+1}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$, so that there exist grΛ-submodules $X, Y \subset \mathrm{Ext}^{s+1}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ such that $U \cong X/Y$. Since $\mathrm{grade}X \geq s+1$ and $\mathrm{grade}Y \geq s+1$ by A.14, it holds that $\mathrm{grade}U \geq s+1$. Therefore, $\mathrm{grade}(\mathrm{Im}\varphi_r) \geq s+1$ for $r \geq 1$. Consider the exact sequence induced from (3):

$$0 \to E^{r+1}_{s+1} \to E^r_{s+1} \to \mathrm{Im}\varphi_r \to 0.$$

Assume that $\mathrm{grade}E^r_{s+1} = s$. Then $\mathrm{grade}E^{r+1}_{s+1} = s$ holds. Hence $\mathrm{grade}E^r_{s+1} = s$ holds for all $r \geq 1$ by induction. Especially, $E^r_{s+1} \neq 0$ holds for all $r \geq 1$.

2.8.5. Lemma *There is an isomorphism* $E^r_{i+1} \cong \mathrm{gr}(\mathrm{Ext}^i_\Lambda(M, \Lambda))$ *for* $i \geq 0$ *and* $r >> 0$.

Proof. Since the filtration $\mathcal{F}$ of Λ is Zariskian (see [14], Chapter I, §2, 2.4; §3, 3.3 and Chapter II, §2, 2.1, and Proposition 2.2.1), the lemma follows from [14], Chapter III, §2, Lemma 2.2.1(p. 150) and §1, Corollary 1.1.7(p. 133). □

2.8.6. We have shown that $E^r_{s+1} \neq 0$. Hence $\mathrm{Ext}^s_\Lambda(M, \Lambda) \neq 0$ by Lemma 2.8.5. Therefore, (i) holds.

Conversely, since $\mathrm{grade}\,\mathrm{gr}M = s$, we have $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda) = 0$ for $i < s$. Since $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda)$ is a subfactor of $\mathrm{Ext}^i_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ by Proposition 2.1, we have $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i < s$. Therefore, $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for $i < s$, so that (ii) holds. This accomplishes the proof of 2.8. □

2.9. Remarks. (i) Let $M \in \mathcal{G}$. Then it follows from 2.3 and 2.8 that

$$\text{G-dim}\,\mathrm{gr}M \geq \text{G-dim}M \geq \mathrm{grade}M = \mathrm{grade}\,\mathrm{gr}M.$$

If $\mathrm{gr}M$ is perfect, then above inequalities are equalities. Hence $M \in \mathcal{G}_e$.

(ii) Let $M \in \mathcal{G}_e$ with $\text{G-dim}M = d$. Then every syzygy $\Omega^i M$ of M is also in $\mathcal{G}_e$. For, as $\mathrm{gr}(\Omega^i M)$ and $\Omega^i(\mathrm{gr}M)$ are stably isomorphic, we see that $\text{G-dim}\,\Omega^i M = \text{G-dim}\,\mathrm{gr}(\Omega^i M) = \max\{0, d-i\}$.

Applying Theorem 2.8 to the case that grΛ is a Gorenstein ring, we get the following.

2.10. Corollary. *Let* Λ *be a filtered ring such that* grΛ *is a commutative Gorenstein ring and* M *a filtered* Λ*-module with a good filtration. Then the equality* $\mathrm{grade}_\Lambda M = \mathrm{grade}_{\mathrm{gr}\Lambda}\mathrm{gr}M$ *holds.*

Proof. Since all the finitely generated grΛ-modules have finite G-dimension (see the proof of [1], Theorem 4.20), this follows from Theorem 2.8. □

2.11. Theorem. *Let* Λ *be a Gorenstein filtered ring. Let* M *be a filtered* Λ*-module with a good filtration. Then the following equality holds.*

$$\mathrm{grade}M + {}^*\dim\mathrm{gr}M = {}^*\dim\mathrm{gr}\Lambda = {}^*\mathrm{id}\,\mathrm{gr}\Lambda.$$

Proof. This follows from A.9, A.10, A.12 and 2.8. □

When Λ is a Gorenstein filtered ring, due to the above equality, we can define a holonomic module. Put ${}^*\mathrm{id}\,\mathrm{gr}\Lambda = n$ and $\mathrm{id}\Lambda = d$. Let M be a filtered Λ-module with a good filtration. Since $\mathrm{grade}M \leq \mathrm{id}\Lambda = d$, we have $n - {}^*\dim\mathrm{gr}M \leq d$, hence

$${}^*\dim\mathrm{gr}M \geq n - d.$$

This inequality is a generalization of Bernstein's inequality for a Weyl algebra ([4]).

According to the case of Weyl algebras, we call a finitely generated filtered Λ-module M a *holonomic module*, if $^*\dim \mathrm{gr}M = n - d$.

3. Cohen-Macaulay modules and holonomic modules

Throughout this section, we assume that Λ is a filtered ring such that $\mathrm{gr}\Lambda$ is a Cohen-Macaulay *local ring with the condition (P) (cf. Appendix). Let M be a finitely generated filtered Λ-module such that $M \in \mathcal{G}$, i.e., $\text{G-dim}\,\mathrm{gr}M < \infty$. It follows from 2.3, 2.8, A.8 and A.12 that the following holds:

$$(1) \quad \text{G-dim}M + {}^*\mathrm{depth}\,\mathrm{gr}M \leq n$$

$$(2) \quad \mathrm{grade}M + {}^*\dim \mathrm{gr}M = n,$$

where we put $n := {}^*\mathrm{depth}\,\mathrm{gr}\Lambda = {}^*\dim \mathrm{gr}\Lambda$. We say that $M \in \mathcal{G}$ is a *Cohen-Macaulay* Λ*-module of codimension* k, if $^*\mathrm{depth}\,\mathrm{gr}M = {}^*\dim \mathrm{gr}M = n - k$. Then it is easily seen that if M is Cohen-Macaulay of codimension k then it is perfect of grade k, where, due to [1], Definition 4.34, we call M perfect if $\text{G-dim}M = \mathrm{grade}M$. Note also that M is Cohen-Macaulay if and only if $\mathrm{gr}M$ is a perfect $\mathrm{gr}\Lambda$-module by A.8 and A.12. We put

$$\mathcal{C}_k(\Lambda) := \{M \in \mathcal{G} : M \text{ is a Cohen-Macaulay } \Lambda\text{-module of codimension } k\}.$$

The following is an easy consequence of (1) and (2).

3.1. Proposition. *Let* $M \in \mathcal{C}_k(\Lambda)$*. Then* $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ *for all* $i \neq k$ $(i \geq 0)$.

We slightly generalize [16], Lemma 2.7 and Theorem 2.8, and [15], as follows.

3.2. Theorem. *Let* $M \in \mathcal{G}$.

i) If $M \in \mathcal{C}_k(\Lambda)$*, then* $\mathrm{Ext}^k_\Lambda(M, \Lambda) \in \mathcal{C}_k(\Lambda^{\mathrm{op}})$.

ii) The functor $\mathrm{Ext}^k_\Lambda(-, \Lambda)$ *induces a duality between the categories* $\underline{\mathcal{C}}_k(\Lambda)$ *and* $\underline{\mathcal{C}}_k(\Lambda^{\mathrm{op}})$.

3.2.1. Lemma. *Let* N *be a finitely generated filtered* Λ*-module of grade* s*. Then we have an embedding* $\mathrm{gr}(\mathrm{Ext}^s_\Lambda(N, \Lambda)) \hookrightarrow \mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda)$*. Moreover, if* $\mathrm{gr}N$ *is perfect, then the embedding is an isomorphism.*

Proof. Let $\cdots \to F_1 \to F_0 \to N \to 0$ be a filtered free resolution of N. We use the notation of 2.8.1. It follows from 2.8.2 and 2.8 that

$$E^1_s \cong H^s(E^0_\bullet) \cong H^{s-1}(F_\bullet) = \mathrm{Ext}^{s-1}_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda) = 0,$$

where a complex $F_\bullet : 0 \to F_0^* \to F_1^* \to \cdots$ is as in 2.8.1. There exists an exact sequence

$$0 \to E^{r+1}_{s+1} \to E^r_{s+1} \to E^r_{s+2}$$

for all $r \geq 1$ by 2.8.3, so that $E^r_{s+1} \subset E^1_{s+1}$ for all $r \geq 1$. It follows from Lemma 2.8.5 that, for $r >> 0$,

$$E^r_{s+1} \cong \mathrm{gr}(\mathrm{Ext}^s_\Lambda(N, \Lambda)).$$

Thus, by 2.8.2, we get

$$\mathrm{gr}(\mathrm{Ext}^s_\Lambda(N, \Lambda)) \subset E^1_{s+1} \cong \mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda).$$

Assume further that $\mathrm{gr}N$ is perfect. Since E^r_{s+2} is a subfactor of $E^1_{s+2} \cong \mathrm{Ext}^{s+1}_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda) = 0$, we see $E^r_{s+2} = 0$, which shows that the embedding is an isomorphism. $\square$

3.2.2. Proof of 3.2. i) Since gr$M$ is perfect of grade k, it holds that $\mathrm{Ext}^k_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}\Lambda)$ is perfect of grade k by [1], Proposition 4.35 and its proof, and so $\mathrm{gr}\,\mathrm{Ext}^k_\Lambda(M,\Lambda)$ is perfect by Lemma 3.2.1. Hence $\mathrm{Ext}^k_\Lambda(M,\Lambda) \in \mathcal{C}_k(\Lambda^{\mathrm{op}})$. ii) Consider the exact sequence

$$0 \to \mathrm{Ext}^k_\Lambda(M,\Lambda) \to \mathrm{Tr}\Omega^{k-1}M \to \Omega\mathrm{Tr}\Omega^k M \to 0$$

(see, for example, the proof of [13], Lemma 2.1) and apply $(-)^*$ to it. Then we get a long exact sequence

$$\mathrm{Ext}^{k+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^k M,\Lambda) \to \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^{k-1}M,\Lambda) \to \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda),\Lambda) \to \mathrm{Ext}^{k+2}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^k M,\Lambda).$$

Since G-dim$\mathrm{Tr}\Omega^k M = 0$ by assumption, the first and fourth terms of the above exact sequence vanishes. Hence $M \cong \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^{k-1}M,\Lambda) \cong \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda),\Lambda)$ by [13], Lemma 2.5. Therefore, there is a natural isomorphism $M \cong \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda),\Lambda)$ for $M \in \mathcal{C}_k(\Lambda)$, which induces a duality between the categories $\underline{\mathcal{C}}_k(\Lambda)$ and $\underline{\mathcal{C}}_k(\Lambda^{\mathrm{op}})$. □

3.2.3. Remark. The proof 3.2.2 ii) only needs M to be perfect with grade$M = k$. Hence we see that if M is perfect of grade k then $M \cong \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Ext}^k_\Lambda(M,\Lambda),\Lambda)$.

We shall study holonomic modules when Λ is a Gorenstein filtered ring, that is, grΛ is Gorenstein, and generalize the former theory which is under the assumption of regularity (cf. [14], Chapter Ⅲ, §4). The assumption that Λ is Gorenstein implies that $\mathcal{G} = \mathrm{fil}\Lambda$ by A.9, where filΛ is the category of all finitely generated filtered (left) Λ-modules. We recall from Corollary 2.4 and the end of section two that $\mathrm{id}_\Lambda\Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}}\Lambda (= d)$ and $M \in \mathrm{fil}\Lambda$ is called holonomic, if ${}^*\mathrm{dim}\,\mathrm{gr}M = n - d$, where $n = {}^*\mathrm{depth}\,\mathrm{gr}\Lambda = {}^*\mathrm{dim}\,\mathrm{gr}\Lambda = {}^*\mathrm{id}\,\mathrm{gr}\Lambda$. We see that if $M \in \mathcal{C}_d(\Lambda)$ then M is holonomic. We also note that M is holonomic if and only if grade$M = d$ (or grade gr$M = d$) if and only if M is perfect of grade d. We keep to assume Λ to be a Gorenstein filtered ring and $d = \mathrm{id}_\Lambda\Lambda$ in the rest of this section.

3.3. Proposition. *Let M be a finitely generated filtered Λ-module. Let M be holonomic and N a Λ-submodule of M. Then N, M/N are holonomic.*

Proof. It follows from [13], Lemma 2.11 (cf. also [6], Theorem 3.9) that grade$N \geq d$ and grade$M/N \geq d$, so that grade$N = d$ and grade$M/N = d$. □

3.4. Proposition. *A holonomic module is artinian. Therefore, it is of finite length.*

We use the following easy lemma for a proof.

3.4.1. Lemma. *Let M_i $(i = 0, 1, \cdots)$ be a module over a ring and $f_i : M_i \to M_{i+1}$ $(i = 0, 1, \cdots)$ is a homomorphism. Assume that M_0 is Noetherian and f_i $(i = 0, 1, \cdots)$ is surjective. Then there exists an interger m such that f_i is an isomorphism for all $i \geq m$.*

3.4.2. Proof of 3.4. Let M be a holonomic Λ-module and $M = M_0 \supset M_1 \supset \cdots$ a descending chain of Λ-submodules of M. Then M_i, M_{i-1}/M_i are holonomic $(i \geq 1)$, and so, from an exact sequence $0 \to M_i \to M_{i-1} \to M_{i-1}/M_i \to 0$, we get an exact sequence

$$0 \to \mathbb{E}(M_{i-1}/M_i) \to \mathbb{E}M_{i-1} \to \mathbb{E}M_i \to 0,$$

where we put $\mathbb{E}(-) = \mathrm{Ext}^d_\Lambda(-,\Lambda)$. By Lemma 3.4.1, there exists an integer m such that $\mathbb{E}M_{i-1} \to \mathbb{E}M_i$ is an isomorphism for $i \geq m+1$. Hence $\mathbb{E}(M_{i-1}/M_i) = 0$ for $i \geq m+1$. Hence $M_{i-1}/M_i = 0$ for $i \geq m+1$ by Remark 3.2.3, that is, $M_m = M_{m+1} = \cdots$. This completes the proof. □

We generalize [14], Chapter Ⅲ, 4.2.18 Theorem (p. 194), which characterizes a holonomic module by its associated graded module. We put $\mathrm{Min}(\mathrm{gr}M) = \{\mathfrak{p} : \mathfrak{p}$ is a minimal element of $\mathrm{Supp}(\mathrm{gr}M)\}$ for $M \in \mathrm{fil}\Lambda$.

3.5. Theorem. *Let $M \in \mathrm{fil}\Lambda$. Then the following are equivalent.*

(1) M is holonomic,

(2) $\mathrm{ht}\mathfrak{p} = d$ *for all* $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$.

A finitely generated module M over a two-sided Noetherian ring is called *pure*, if $\mathrm{grade}N = \mathrm{grade}M$ for all nonzero submodules N of M.

3.5.1. Lemma. *Let $M \in \mathrm{fil}\Lambda$. Then M is pure if and only if* $\mathrm{gr}M$ *is a pure* $\mathrm{gr}\Lambda$*-module under a suitable filtration on M.*

Proof. Let M be pure. Put $s = \mathrm{grade}M$ and $N := \mathrm{Ext}^s_\Lambda(M, \Lambda)$. Then $\mathrm{grade}\,\mathrm{gr}N = \mathrm{grade}N = s$ by 2.8 and A.15, hence $\mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda)$ is pure by [13], Proposition 2.13. By 3.2.1, we have $\mathrm{gr}\,\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda) \subset \mathrm{Ext}^s_{\mathrm{gr}\Lambda}(\mathrm{gr}N, \mathrm{gr}\Lambda)$. Hence $\mathrm{gr}\,\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$ is a pure $\mathrm{gr}\Lambda$-module. By [13], Theorem 2.3, there exists an exact sequence

$$0 \to \mathrm{Ext}^{s+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^s M, \Lambda) \to M \to \mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda).$$

Since $\mathrm{grade}\,\mathrm{Ext}^{s+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^s M, \Lambda) \geq s + 1$ and M is pure, we see $\mathrm{Ext}^{s+1}_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}\Omega^s M, \Lambda) = 0$. Therefore, $M \subset \mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$. According to a filtration on M induced from that of $\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$, we get an inclusion $\mathrm{gr}M \subset \mathrm{gr}\,\mathrm{Ext}^s_{\Lambda^{\mathrm{op}}}(N, \Lambda)$, hence $\mathrm{gr}M$ is pure. The converse is obvious by Theorem 2.8. □

3.5.2. Lemma. *Let R be a commutative Gorenstein ring and M' a pure R-module. Then* $\mathrm{grade}M' = \mathrm{dim}R_\mathfrak{p}$ *for each* $\mathfrak{p} \in \mathrm{Min}(M')$.

Proof. Since $R_\mathfrak{p}$ is a Gorenstein local ring, we have an equality $\mathrm{grade}M'_\mathfrak{p} + \mathrm{dim}M'_\mathfrak{p} = \mathrm{dim}R_\mathfrak{p}$ (cf. [11], Proposition 4.11). Since $\mathfrak{p}$ is minimal, $\mathrm{dim}M'_\mathfrak{p} = 0$, so that, $\mathrm{grade}M'_\mathfrak{p} = \mathrm{dim}R_\mathfrak{p}$.

Put $g = \mathrm{grade}M'_\mathfrak{p}$, $g' = \mathrm{grade}M'$. Since $\mathrm{Ext}^g_R(M', R)_\mathfrak{p} = \mathrm{Ext}^g_{R_\mathfrak{p}}(M'_\mathfrak{p}, R_\mathfrak{p}) \neq 0$, we have $\mathrm{Ext}^g_R(M', R) \neq 0$. Hence $g \geq g'$ holds. Suppose that $\mathrm{Ext}^k_R(\mathrm{Ext}^k_R(M', R), R) \neq 0$ for $k > g'$. Then there exists $N \subset M'$ such that $\mathrm{grade}N = k > g'$ by [13], Theorem 2.3, which contradicts the purity of M'. Hence $\mathrm{Ext}^k_R(\mathrm{Ext}^k_R(M', R), R) = 0$ for all $k > g'$. But by A.15, $\mathrm{grade}\,\mathrm{Ext}^g_{R_\mathfrak{p}}(M'_\mathfrak{p}, R_\mathfrak{p}) = g$. Therefore, we see $g \leq g'$, and so, $g = g'$. This completes the proof. □

3.5.3. proof of Theorem 3.5. Put $R = \mathrm{gr}\Lambda$.

(1)⇒(2): Assume that M is holonomic. Since M is pure by Proposition 3.3, $\mathrm{gr}M$ is pure by 3.5.1. Thus $d = \mathrm{grade}\,\mathrm{gr}M = \mathrm{dim}R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$ by 3.5.2. Therefore, $\mathrm{ht}\mathfrak{p} = d$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$.

(2)⇒(1): Put $I = [0 :_R \mathrm{gr}M]$. Since R is Cohen-Macaulay, we have $\mathrm{ht}I = \mathrm{grade}R/I$ by [8], Corollary 2.1.4. It follows from [8], Proposition 1.2.10(e) that $\mathrm{grade}R/I = \mathrm{grade}\,\mathrm{gr}M$. By assumption, $\mathrm{ht}I = d$, so that, $\mathrm{grade}\,\mathrm{gr}M = d$, that is, $\mathrm{grade}M = d$ by 2.8. Hence M is holonomic. □

A module having higher grade has a good Ext-group.

3.6. Proposition. *Let M be a finitely generated filtered Λ-module with* $\mathrm{grade}M = \ell$, *where $\ell = d, d-1$ or $d-2$. Then M is a perfect Λ-module if and only if there exists a finitely generated filtered Λ-module M' of grade ℓ with* $M \cong \mathrm{Ext}^\ell_\Lambda(M', \Lambda)$.

Proof. When $\ell = d$, M, M' are holonomic, so the equivalence is obvious. We assume that $M \cong \mathrm{Ext}^{\ell}_{\Lambda}(M', \Lambda)$ with $\mathrm{grade} M' = \ell$.

The case $\ell = d-1$: It follows that $\mathrm{grade}\mathrm{Ext}^d_{\Lambda}(M, \Lambda) = \mathrm{grade}\mathrm{Ext}^d_{\Lambda}(\mathrm{Ext}^{d-1}_{\Lambda}(M', \Lambda), \Lambda) \geq d+2$ by [13], Corollary 2.10. This shows that $\mathrm{Ext}^d_{\Lambda}(M, \Lambda) = 0$, that is, $\text{G-dim}M \leq d-1$. Hence $\text{G-dim}M = \mathrm{grade}M = d-1$.

The case $\ell = d-2$: It follows from the similar computations as the above case that $\mathrm{grade}\mathrm{Ext}^d_{\Lambda}(M, \Lambda) \geq d+2$ and $\mathrm{grade}\mathrm{Ext}^{d-1}_{\Lambda}(M, \Lambda) \geq d+1$. Hense $\mathrm{Ext}^d_{\Lambda}(M, \Lambda) = \mathrm{Ext}^{d-1}_{\Lambda}(M, \Lambda) = 0$, so that $\text{G-dim}M = \mathrm{grade}M = d-2$.

Since the converse is clear, this completes the proof. □

3.7. Following [14], Chapter Ⅲ, 4.3, we call a filtered Λ-module M *geometrically pure* (*geo-pure* for short), if $\dim_{\mathrm{gr}\Lambda}\mathrm{gr}M = \dim(\mathrm{gr}\Lambda/\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$. Then we have the following proposition which is a generalization of [14], Chapter Ⅲ, 4.3.6 Corollary.

3.7.1. Proposition. *Let M be a finitely generated filtered Λ-module. Then the following conditions are equivalent.*

(1) M is pure,

(2) M is geo-pure and $\mathrm{gr}M$ *has no embedded prime.*

Proof. (1)⇒(2): Let M be pure. Then $\mathrm{gr}M$ is pure by 3.5.1. Take any $\mathfrak{p} \in \mathrm{Min}(\mathrm{gr}M)$. Since $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{gr}M$, we have $\mathrm{gr}\Lambda/\mathfrak{p} \hookrightarrow \mathrm{gr}M$, so $\mathrm{grade}\,\mathrm{gr}\Lambda/\mathfrak{p} = \mathrm{grade}\,\mathrm{gr}M$. Using Theorem A.12, we have $\dim \mathrm{gr}\Lambda/\mathfrak{p} = \dim \mathrm{gr}M$. Hence M is geo-pure. Take any $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{gr}M$, then $\mathrm{gr}\Lambda/\mathfrak{p} \hookrightarrow \mathrm{gr}M$. Thus $\dim \mathrm{gr}\Lambda/\mathfrak{p} = \dim \mathrm{gr}M$, by A.12. Therefore, $\mathrm{Ass}\,\mathrm{gr}M = \mathrm{Min}\,\mathrm{gr}M$, i.e., $\mathrm{gr}M$ has no embedded primes.

(2)⇒(1): By 3.5.1, it suffices to prove that $\mathrm{gr}M$ is pure. Let N be a $\mathrm{gr}\Lambda$-submodule of $\mathrm{gr}M$. Take any $\mathfrak{p} \in \mathrm{Ass}N$. Then $\mathrm{gr}\Lambda/\mathfrak{p} \hookrightarrow N$. It follows that $\mathfrak{p} \in \mathrm{Ass}\,\mathrm{gr}M = \mathrm{Min}\,\mathrm{gr}M$ by assumption. Thus, by A.12 and assumption, we have $\mathrm{grade}\,\mathrm{gr}\Lambda/\mathfrak{p} = \mathrm{grade}\,\mathrm{gr}M$. By [13], Lemma 2.11, we have

$$\mathrm{grade}\,\mathrm{gr}M \leq \mathrm{grade}N \leq \mathrm{grade}\,\mathrm{gr}\Lambda/\mathfrak{p} = \mathrm{grade}\,\mathrm{gr}M.$$

Hence $\mathrm{grade}\,\mathrm{gr}M = \mathrm{grade}N$. This completes the proof. □

3.8. Example. We provide an example of a Gorenstein filtered ring Λ.
Let $R = k[[x^2, x^3]]$ be a subring of a formal power series ring $k[[x]]$, where k is a field of characteristic zero. Then $(R, \mathfrak{m})$ is a local Gorenstein (non-regular) ring of $\dim R = 1$, where $\mathfrak{m} = (x^2, x^3)$. Let a differential operator $T = x\partial$ with $\partial = d/dx$. Let Λ be a subring of the first Weyl algebra (see [4], [14]) generated by R and T. Then every element of Λ is written as $\Sigma a_i T^i$, $a_i \in R$. Note that $Tx^i = x^i T + ix^i$, $i \geq 2$. For $P = \Sigma a_i T^i \in \Lambda$, we put $\mathrm{ord}P = \max\{i : a_i \neq 0\}$, an order of P. Let $\mathcal{F}_i\Lambda := \{P \in \Lambda : \mathrm{ord}P \leq i\}$. Then $\{\mathcal{F}_i\Lambda\}$ is a filtration of Λ and $\mathrm{gr}\Lambda = R[t]$, where $t = \sigma_1(T)$. Thus $\mathrm{gr}\Lambda$ is Gorenstein *local of dimension 2. Note that $\mathfrak{m} + tR[t]$ is a unique *maximal ideal.

1) $\mathrm{id}\Lambda = 2$
Let $I := \Lambda T + \Lambda x^2$ be a left ideal of Λ. Then $I \neq \Lambda$. We put induced filtrations to I and Λ/I. , i.e.,

$$\mathcal{F}_i I = I \cap \mathcal{F}_i\Lambda, \quad \mathcal{F}_i(\Lambda/I) = (\mathcal{F}_i\Lambda + I)/I, \quad i \geq 0.$$

Then $0 \to I \to \Lambda \to \Lambda/I \to 0$ is a strict exact sequence. Hence $0 \to \mathrm{gr}I \to \mathrm{gr}\Lambda \to \mathrm{gr}(\Lambda/I) \to 0$ is exact. Since $\mathrm{gr}I$ contains t and x^2, $\mathrm{gr}(\Lambda/I) = \mathrm{gr}\Lambda/\mathrm{gr}I$ is an Artinian

$\mathrm{gr}\Lambda$-module. Hence $\dim_{\mathrm{gr}\Lambda}\mathrm{gr}(\Lambda/I) = 0$. Thus $\mathrm{grade}\,\Lambda/I = 2$ by Theorem 2.11, and then $\mathrm{id}\Lambda = 2$ by Corollary 2.4.

2) $\mathrm{gl}\dim\Lambda = \infty$
It is easily seen that $\mathrm{gr}(\Lambda/\Lambda\mathfrak{m}) \cong R/\mathfrak{m}[t]$, where a filtration of $\Lambda\mathfrak{m}$ is given by $\mathcal{F}_i(\Lambda\mathfrak{m}) = (\mathcal{F}_i\Lambda)\mathfrak{m}$. Hence $\mathrm{pd}_{\mathrm{gr}\Lambda}\mathrm{gr}(\Lambda/\Lambda\mathfrak{m}) = \infty$ which implies $\mathrm{pd}_\Lambda\Lambda/\Lambda\mathfrak{m} = \infty$ by Remark 2.7 (ii). Hence $\mathrm{gl}\dim\Lambda = \infty$.

3) A filtered Λ-module M with a good filtration is holonomic if and only if $\mathrm{grade}M = 2$. Since $\mathrm{gr}(\Lambda/I)$ is Cohen-Macaulay of codimension two, we see that $\text{G-dim}\,\mathrm{gr}(\Lambda/I) = \mathrm{grade}\,\mathrm{gr}(\Lambda/I) = 2$. Hence $\text{G-dim}\Lambda/I = \mathrm{grade}\Lambda/I = 2$ by Remark 2.9 (i), so that Λ/I is holonomic. On the other hand, since $\mathrm{gr}(\Lambda/\Lambda\mathfrak{m})$ is Cohen-Macaulay of codimension one, a Λ-module $\Lambda/\Lambda\mathfrak{m}$ is not holonomic.

Appendix

In Appendix, we provide the fact about graded rings, especially *local rings .

1. Summary for *local rings

Let R be a commutative Noetherian ring. We gather some facts about a graded ring. For the detail, the reader is referred to [8], [12], and [20].

A ring R is called a *graded ring*, if

i) $R = \oplus_{i\in\mathbb{Z}}R_i$ as an additive group,

ii) $R_iR_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$.

An R-module M is called a *graded module*, if

i) $M = \oplus_{i\in\mathbb{Z}}M_i$ as an additive groups,

ii) $R_iM_j \subset M_{i+j}$ for all $i,\ j \in \mathbb{Z}$.

An R-homomorphism $f : M \to N$ of graded modules is called a *graded homomorphism*, if $f(M_i) \subset N_i$ for all $i \in \mathbb{Z}$. All graded modules in $\mathrm{mod}R$ and all graded homomorphisms form the category of graded modules, which we denote by mod_0R.

A graded submodule of a graded ring R is called a *graded ideal*. For any ideal I of R, we denote by I^* the graded ideal generated by all homogeneous elements of I. A graded ideal $\mathfrak{m}$ of R is called **maximal*, if it is a maximal element of all proper graded ideals of R. We say that R is a **local* ring, if R has a unique *maximal ideal $\mathfrak{m}$. A *local ring R with the *maximal ideal $\mathfrak{m}$ is denoted by $(R, \mathfrak{m})$. The theory of *local ring is well developed and a lot of facts that hold for local rings also hold for *local rings (see [8] and [12]).

Let M be a finite R-module. For an ideal I, we denote I-depth of M by $\mathrm{depth}(I, M)$([18]). Let $(R, \mathfrak{m})$ be a *local ring and $M \in \mathrm{mod}R$. We put ${}^*\mathrm{depth}M := \mathrm{depth}(\mathfrak{m}, M)$. We shall use *depth as a substitute of depth for a local ring.

A graded module M over a graded ring R is called a **injective* module, if it is an injective object in mod_0R([8], §3.6). We denote by ${}^*\mathrm{id}M$ the *injective dimension of M. By definition, ${}^*\mathrm{id}M \le k$ if and only if there exists a minimal *injective resolution

$$0 \to M \to {}^*\!E^0(M) \to \cdots \to {}^*\!E^k(M) \to 0.$$

It is easily seen that ${}^*\mathrm{id}M \le k$ if and only if $\mathrm{Ext}^i_R(N, M) = 0$ for all $i > k$ and all $N \in \mathrm{mod}_0R$.

Let $(R, \mathfrak{m})$ be a *local ring. Consider the following condition.

(P) There exists an element of positive degree in $R - \mathfrak{p}$
for any graded prime ideal $\mathfrak{p} \neq \mathfrak{m}$

A positively graded ring satisfies the condition (P). The other examples are seen in [20], Chapter B, Ⅲ, 3.2.

The following is known.

A.1. Proposition. *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P). Then, for every graded ideal $\mathfrak{a}$ and every set of graded prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$, there exists i such that $\mathfrak{a} \subset \mathfrak{p}_i$, whenever all homogeneous elements of $\mathfrak{a}$ are contained in $\cup_{i=1}^{n} \mathfrak{p}_i$.*

Proof. See [19], Lemma 2. □

Using Proposition A.1, the following is proved as the local case.

A.2. Proposition. *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P). Let M be a finite graded module with* $^*\mathrm{depth} M = t$. *Then there exists an M-sequence $x_1, \cdots, x_t$ consisting of homogeneous elements in $\mathfrak{m}$.*

We note the following graded version of Nakayama's Lemma.

A.3. Lemma. *Let $(R, \mathfrak{m})$ be a *local ring and M a finite graded R-module. If $\mathfrak{m}M = M$, then $M = 0$.*

In the following, we assume that $(R, \mathfrak{m})$ is a *local ring with the condition (P).

A.4. Lemma. *Let M, N be the non-zero finite graded R-module with* $^*\mathrm{depth} N = 0$. *Then* $\mathrm{Hom}_R(M, N) \neq 0$.

Proof. It is well-known, so we omit the proof. □

A.5. Corollary. *Assume that* $^*\mathrm{depth} R = 0$. *Let M be a finite graded R-module. Then $M^* = 0$ implies $M = 0$.*

We state the graded version of [1], 4.11-13 in the following A.6-A.8.

A.6. Proposition. *Assume that* $^*\mathrm{depth} R = 0$. *Let M be a finite graded R-module. Then* $\text{G-dim} M < \infty$ *if and only if* $\text{G-dim} M = 0$.

Proof. It suffices to prove that $\text{G-dim} M < \infty$ implies $\text{G-dim} M = 0$.

Suppose that $\text{G-dim} M \leq 1$. We have an exact sequence $0 \to L_1 \to L_0 \to M \to 0$ with $\text{G-dim} L_i = 0$ $(i = 0, 1)$. Hence we have an exact sequence

$$0 \to M^* \to L_0^* \to L_1^* \to \mathrm{Ext}_R^1(M, R) \to 0$$

and $\mathrm{Ext}_R^i(M, R) = 0$ for $i > 1$. By this sequence, we have an exact sequence

$$0 \to \mathrm{Ext}_R^1(M, R)^* \to L_1 \to L_0,$$

where $L_1 \to L_0$ is monic. Thus $\mathrm{Ext}_R^1(M, R)^* = 0$, and so $\mathrm{Ext}_R^1(M, R) = 0$ by Corollary 2.5.

Suppose that $\text{G-dim} M \leq n$. Let $0 \to L_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} L_0 \to M \to 0$ be exact with $\text{G-dim} L_i = 0$ $(0 \leq i \leq n)$. Since $\text{G-dim}(\mathrm{Im} f_{n-1}) \leq 1$, we have $\text{G-dim}(\mathrm{Im} f_{n-1}) = 0$ by the above argument. Repeating this process, we get $\text{G-dim} M = 0$. □

We want to generalize [1], Theorem 4.13 (b) to the graded case. The proof of it needs a part of [1], Proposition 4.12. Thus we adapt this proposition as follows.

A.7. Proposition. *Assume that* ${}^*\mathrm{depth}R = t$. *Let* M *be a finite graded* R*-module with* $\text{G-dim}M < \infty$. *Then the following are equivalent.*

(1) $\text{G-dim}M = 0$.

(2) ${}^*\mathrm{depth}M \geq {}^*\mathrm{depth}R$.

(3) ${}^*\mathrm{depth}M = {}^*\mathrm{depth}R$.

Proof. (1) $\Rightarrow$ (2): Let $x_1, \cdots, x_i$ be a homogeneous regular sequence in $\mathfrak{m}$. We show that $x_1, \cdots, x_i$ is an M-sequence by induction on i. Let $i = 1$. Since $M \cong M^{**}$ is torsionfree, x_1 is M-regular.

Suppose that $i > 1$ and the assertion holds for $i-1$. Then $x_1, \cdots, x_{i-1}$ is an M-sequence. Put $I = (x_1, \cdots, x_i)$, $\overline{R} = R/I$, $\overline{M} = M/IM$. Then $(\overline{R}, \mathfrak{m}/I)$ is a *local ring with the condition (P). By [1], Lemma 4.9, $\text{G-dim}_{\overline{R}}\overline{M} = \text{G-dim}_R M = 0$. Since $\overline{x}_i \in \overline{R}$ is a regular element, $\overline{x}_i$ is $\overline{M}$-regular, hence $x_1, \cdots, x_i$ is an M-sequence. Therefore, ${}^*\mathrm{depth}M \geq {}^*\mathrm{depth}R$.

(2) $\Rightarrow$ (1): By assumption, it suffices to prove that $\mathrm{Ext}^i_R(M, R) = 0$ for $i > 0$. We show the assertion by induction on $t = {}^*\mathrm{depth}R$.

Let $t = 0$. Then $\text{G-dim}M = 0$ by Proposition A.6

Let $t > 0$. Then ${}^*\mathrm{depth}M \geq {}^*\mathrm{depth}R \geq 1$. We take a homogeneous element $x \in \mathfrak{m}$ which is R and M-regular. Then, by [8], 1.2.10 (d),

$${}^*\mathrm{depth}_{R/xR}M/xM = {}^*\mathrm{depth}_R M - 1 \geq {}^*\mathrm{depth}R - 1 = {}^*\mathrm{depth}R/xR.$$

Hence we have $\mathrm{Ext}^i_{R/xR}(M/xM, R/xR) = 0$ for $i > 0$ by induction. This gives $\mathrm{Ext}^i_R(M, R/xR) = 0$ for $i > 0$. From an exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$, we get an exact sequence

$$\mathrm{Ext}^i_R(M, R) \xrightarrow{x} \mathrm{Ext}^i_R(M, R) \to \mathrm{Ext}^i_R(M, R/xR) = 0.$$

By Nakayama's Lemma, it holds that $\mathrm{Ext}^i_R(M, R) = 0$ for $i > 0$.

(2) $\Rightarrow$ (3): When ${}^*\mathrm{depth}R = 0$, we have $\text{G-dim}M = 0$ by Proposition A.6. Since ${}^*\mathrm{depth}R = 0$, we have an exact sequence $0 \to R/\mathfrak{m} \to R$ which gives an exact sequence

$$0 \to \mathrm{Hom}_R(M^*, R/\mathfrak{m}) \to M^{**} \cong M.$$

Since $M^* \neq 0$, we have $\mathrm{Hom}_R(M^*, R/\mathfrak{m}) \neq 0$. Since $\mathfrak{m}\mathrm{Hom}_R(M^*, R/\mathfrak{m}) = 0$, we see that $\mathfrak{m}$ has no M-regular element, so that ${}^*\mathrm{depth}M = 0$. Thus (3) holds.

Let ${}^*\mathrm{depth}R > 0$. We have ${}^*\mathrm{depth}M \geq {}^*\mathrm{depth}R \geq 1$, so that there is a homogeneous element $x \in \mathfrak{m}$ which is R and M-regular. By [1], Lemma 4.9, we have $\text{G-dim}_{R/xR}M/xM < \infty$. We have

$${}^*\mathrm{depth}_{R/xR}M/xM = {}^*\mathrm{depth}_R M - 1 \geq {}^*\mathrm{depth}R - 1 = {}^*\mathrm{depth}R/xR.$$

Hence, by induction on ${}^*\mathrm{depth}R$, we have ${}^*\mathrm{depth}_{R/xR}M/xM = {}^*\mathrm{depth}R/xR$, and then ${}^*\mathrm{depth}M = {}^*\mathrm{depth}R$.

Since (3) $\Rightarrow$ (2) is obvious, we accomplish the proof. □

A.8. Theorem. *Let* M *be a finite graded* R*-module with* $\text{G-dim}M < \infty$. *Then we have an equality*

$$\text{G-dim}M + {}^*\mathrm{depth}M = {}^*\mathrm{depth}R$$

Proof. We state the proof which is an adaptation of [1]. If $\text{G-dim}M = 0$, we are done by the previous proposition. Suppose that $\text{G-dim}M = n > 0$ and the equation holds for $n-1$. Let $0 \to K \to F \to M \to 0$ be exact with F graded free and K a graded module. Since G-dim$K = n-1$, we have G-dim$K$ + *depth$K$ = *depth$R$ by induction. Suppose that *depth$M \geq {}^*$depth$F = {}^*$depth$R$. Then G-dim$M = 0$ holds by the previous proposition. This contradicts to G-dim$M > 0$. Hence *depth$M < {}^*$depth$F$, so *depth$K = {}^*$depth$M + 1$ by, e.g., [8], 1.2.9. Therefore, $n + {}^*$depth$M = {}^*$depth$R$. □

Let M be a finite graded R-module. Then the similar argument to [1], 4.14 and 4.15 shows that G-dim$M \leq n$ if and only if G-dim$M_{\mathfrak{p}} \leq n$ for all graded prime (respectively, graded maximal) ideals $\mathfrak{p}$ of R. Note that all the prime ideals in Ass$M$ are graded ideals (e.g. [8], Lemma 1.5.6). Thus, in *local case, we have that G-dim$M \leq n$ if and only if G-dim$M_{\mathfrak{m}} \leq n$. Thus we give the following characterization of Gorensteiness.

A.9. Theorem. *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P). Then the following are equivalent.*

(1) R is Gorenstein.

(2) Every finite graded R-module has finite G-dimension.

Under these equivalent conditions, the equality *id$R = {}^*$depth$R$ *holds.*

Proof. (1) ⇒ (2): Since $R_{\mathfrak{m}}$ is Gorenstein, we have G-dim$M_{\mathfrak{m}} < \infty$, hence G-dim$M < \infty$ by above.

(2) ⇒ (1): Let $t = {}^*$depth$R$. Take any finite graded R-module M. Since G-dim$M = t - {}^*$depth$M \leq t$ by Theorem A.8, we have that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > t$. Hence *id$R \leq t$. It holds from [8], Theorem 3.6.5 or [20], Chapter B, III.1.7 that id$R \leq {}^*$id$R + 1 \leq t + 1$. Hence R is Gorenstein.

The second statement follows from the similar argument to the local case (cf. [8],Theorem 3.1.17). We note that 'the residue field' in the local case should be replaced by 'the unique graded simple module $R/\mathfrak{m}$' in *local case and the use of the graded version of Bass's Lemma (see e.g. [20], Chapter B, III.1.9) is effective. □

Let $(R, \mathfrak{m})$ be a *local ring. Then one of the following cases occurs ([12], §1 or [8], §1.5):

A. $R/\mathfrak{m}$ is a field,

B. $R/\mathfrak{m} \cong k[t, t^{-1}]$, where k is a field and t is a homogeneous element of positive degree and transcendental over k.

We put *dim$R :=$ ht$\mathfrak{m}$ the *dimension of a *local ring $(R, \mathfrak{m})$. Note that *dim$R$ equals the supremum of all numbers h such that there exists a chain of graded prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_h$ in R [8]. Let M be a finite graded prime ideal. It is easily seen that $[0 :_R M]$ is a graded ideal. Thus we put *dim$M := {}^*$dim$R/[0 :_R M]$.

A.10. Lemma. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with the condition (P) and* dim$R = n$*, and M a finite graded R-module. Then we have*

$$ {}^*\mathrm{dim}R = {}^*\mathrm{depth}R = \begin{cases} n & \text{for Case A,} \\ n-1 & \text{for Case B.} \end{cases} $$

$$ {}^*\mathrm{dim}M = \begin{cases} \mathrm{dim}M & \text{for Case A,} \\ \mathrm{dim}M - 1 & \text{for Case B.} \end{cases} $$

Moreover, assume that R is Gorenstein, then id$R$ = dim$R = n$*, where* id$R$ *stands for the injective dimension of R.*

Proof. Case A. Let $\mathfrak{n}$ be a maximal ideal with ht $\mathfrak{n} = n$. If $\mathfrak{n} = \mathfrak{m}$, then ht $\mathfrak{m} = n$. Suppose that $\mathfrak{n}$ is not equal to $\mathfrak{m}$. Then $\mathfrak{n}$ is not graded, so ht $\mathfrak{n}/\mathfrak{n}^* = 1$. Since $R_\mathfrak{n}$ is Cohen-Macaulay,

$$\text{ht } \mathfrak{n}^* R_\mathfrak{n} + \dim R_\mathfrak{n}/\mathfrak{n}^* R_\mathfrak{n} = \dim R_\mathfrak{n} = n$$

([18], Theorem 17.4). Hence ht $\mathfrak{n}^* R_\mathfrak{n} = n-1$, so ht $\mathfrak{n}^* = n-1$. Thus ht $\mathfrak{m} \geq$ ht $\mathfrak{n}^*+1 = n$, so that ht $\mathfrak{m} = n$. Therefore,

$$^*\text{depth} R = \text{depth} R_\mathfrak{m} = \dim R_\mathfrak{m} = \text{ht } \mathfrak{m} = n.$$

Case B. Let $\mathfrak{n}$ be the same as in Case A. Since $\mathfrak{n}$ is not graded, we have ht $\mathfrak{n}^* = n-1$ by the similar way to Case A. By assumption, we have that $\mathfrak{m} \supset \mathfrak{n}^*$ and $\mathfrak{m}$ is not maximal, so $\mathfrak{m} = \mathfrak{n}^*$. Therefore, ht $\mathfrak{m} = n-1$, hence we get $^*\text{depth} R = n-1$ by the similar way to Case A.

The equality concerning $^*\dim M$ follows from the fact that cases A and B are preserved modulo $[0 :_R M]$.

The latter statement is proved in [3] more generally. □

A.11. Lemma *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with the condition (P) and x a homogeneous element in $\mathfrak{m}$. If x is regular, then* $\dim R/xR = \dim R - 1$.

Proof. The well-known induction argument works due to Lemma 2.10. □

A.12. Theorem *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with the condition (P) and M a finite graded R-module. Then*

$$\text{grade} M + \dim M = \dim R$$

Proof. We follow the proof of [11], Proposition 4.11. Put $n = \dim R$. We prove the statement by induction on n. Suppose that $\dim M = n$ and take $\mathfrak{p} \in \text{Supp} M$ with $\dim R/\mathfrak{p} = n$. Then $\dim R_\mathfrak{p} = 0$, so that $\text{depth} R_\mathfrak{p} = 0$. Thus $\mathfrak{p} R_\mathfrak{p} \in \text{Ass} R_\mathfrak{p}$. Hence $\text{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \neq 0$ implies $\text{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$. Thus $\text{Hom}_R(M, R) \neq 0$, i.e., $\text{grade} M = 0$.

When $n = 0$, we have $\dim M = 0$. Then the equality holds by above. Let $n > 0$. Then we can assume $\dim M < n$. Since $\dim R/\mathfrak{p} = n$ for any minimal prime ideal $\mathfrak{p}$ of R, it holds from the assumption that $[0 :_R M] \not\subset \mathfrak{p}$ for any minimal prime ideal $\mathfrak{p}$ of R. Thus $[0 :_R M] \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in \text{Ass} R$. Since $[0 :_R M]$ is a graded ideal, $[0 :_R M]$ contains a homogeneous regular element x by Proposition 2.1. We have that $\text{Ext}^i_R(M, R) \cong \text{Ext}^{i-1}_{R/xR}(M, R/xR)$ for $i \geq 0$. Thus $\text{grade}_{R/xR} M = \text{grade}_R M - 1$. By Lemma A.11 and induction, we get $\dim_{R/xR} M + \text{grade}_{R/xR} M = n-1$, hence $\dim_R M + \text{grade}_R M - 1 = n - 1$, which gives the desired equality. □

We state a characterization of a Cohen-Macaulay graded module over a *local ring by means of the *depth and *dimension.

A.13. Theorem *Let $(R, \mathfrak{m})$ be a *local ring with the condition (P) and $M \in \text{mod}_0 R$. Then M is Cohen-Macaulay if and only if $^*\text{depth} M = {}^*\dim M$.*

Proof. Put $I = [0 :_R M]$ and $\overline{R} = R/I$, $\overline{\mathfrak{m}} = \mathfrak{m}/I$. Then we have that $^*\dim M = \dim \overline{R}_{\overline{\mathfrak{m}}} = \dim R_\mathfrak{m}/[0 :_{R_\mathfrak{m}} M_\mathfrak{m}] = \dim M_\mathfrak{m}$. It holds from [19] or [20], Chapter B, Theorem III.2.1 that M is Cohen-Macaulay if and only if $M_\mathfrak{m}$ is Cohen-Macaulay. Look at the following inequalities

$$^*\text{depth} M = \text{depth}(\mathfrak{m}, M) \leq \text{depth} M_\mathfrak{m} \leq \dim M_\mathfrak{m} = {}^*\dim M.$$

If ${}^*\mathrm{depth}M = {}^*\mathrm{dim}M$, then $M_\mathfrak{m}$ is Cohen-Macaulay by above. Conversely, suppose M to be Cohen-Macaulay. Then $\mathrm{depth}(\mathfrak{m}, M) = \mathrm{depth}M_\mathfrak{m}$ holds by [18], Theorem 17.3. Thus we get ${}^*\mathrm{depth}M = {}^*\mathrm{dim}M$ from the above inequalities. □

A.14. Lemma. ([1], Proposition 4.16) *Let R be a commutative Noetherian ring and X a finite R-module with* $\text{G-dim}X < \infty$. *Then* $\mathrm{grade}U \geq i$ *for all* $i > 0$ *and all R-submodules U of* $\mathrm{Ext}^i_R(X, R)$.

Proof. Let $\mathfrak{p} \in \mathrm{Supp}U$. Then $\mathrm{Ext}^i_{R_\mathfrak{p}}(X_\mathfrak{p}, R_\mathfrak{p}) \neq 0$. Hence $\text{G-dim}_{R_\mathfrak{p}} X_\mathfrak{p} \geq i$. By Auslander-Bridger formula ([1], Theorem 4.13 (b) or [9], Theorem 1.4.8), it follows that

$$\mathrm{depth}R_\mathfrak{p} = \mathrm{depth}X_\mathfrak{p} + \text{G-dim}_{R_\mathfrak{p}} X_\mathfrak{p} \geq \text{G-dim}_{R_\mathfrak{p}} X_\mathfrak{p} \geq i.$$

Hence $\mathrm{grade}U = \min\{\mathrm{depth}R_\mathfrak{p} : \mathfrak{p} \in \mathrm{Supp}U\} \geq i$ by [1], Corollary 4.6. □

A.15. Lemma. *Let R be a commutative Noetherian ring and X a finite R-module of grade s. Assume* $\text{G-dim}X$ *to be finite. Then the equality* $\mathrm{grade}\,\mathrm{Ext}^s_R(X, R) = s$ *holds true.*

Proof. When $s = 0$, that is, $X^* \neq 0$, then $X^{***} \neq 0$. Hence $X^{**} \neq 0$.

We assume that $s > 0$. By A.14, it holds that $\mathrm{grade}\,\mathrm{Ext}^s_R(X, R) \geq s$. The converse inequality follows from [13], Lemma 4.4 (Its proof contains trivial misprints : in the last line of p.182, X^*_n should be read $(\Omega^n X)^*$ and three places in line 3-5 of p.183 should be read similarly). Hence we get the desired equality. □

References

[1] M. Auslander and M. Bridger, *Stable module theory*, Mem. of the AMS 94, Amer. Math. Soc., Providence 1969.

[2] M. Auslander, I. Reiten, and S. O. Smalo,*Representation Theory of Artin Algebras*, Cambridge stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. 82 (1963) 8-28.

[4] J-E. Björk, *Rings of Differential Operators*, Math. Ligrary 21, North Holland, 1979.

[5] J-E. Björk, *Filtered Noetherian rings*, in Noetherian Rings and Their Applications Ed. L. W. Small, AMS Math, Surveys and Monographs, vol.24, 1987, 59-97.

[6] J-E. Björk, *The Auslander condition on Noetherian rings*, in Sém. d'Algèbre P. Dubreil et M.-P. Malliavin, 1987-88 (M.-P. Malliavin, ed.), Lecture Notes in Math. 1404, Springer, 1989, 137-173.

[7] J-E. Björk and E. K. Ekström, *Auslander Gorenstein rings*, in Operator algebras, unitary representations, enveloping algebras, and invariant theory, Progr. Math. 92, Birkhauser (1990) 425-448.

[8] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math. 39, Cambridge Univ. Press, 1993.

[9] L. W. Christensen, *Gorenstein Dimensions*, Lecture Notes in Mathematics 1747, Springer, 2000.

[10] C. Faith, *Algebra* Ⅱ *Ring Theory*, Springer, 1976.

[11] S. Goto and K. Nishida, *Towards a theory of Bass numbers with application to Gorenstein algebras*, Colloq. Math. 91(2) (2002) 191-253.

[12] S. Goto and K. Watanabe, *On graded rings,* I, J. Math. Soc. Japan, 30 (1978) 179-213.

[13] M. Hoshino and K. Nishida, *A generalization of the Auslander Formula*, Representations of Algebras and Related Topics, Fields Institute Communications vol. 45 (2005) 175-186.

[14] L. Huishi and F. Van Oystaeyen, *Zariskian Filtrations*, K-Monographs in Mathematics, 2, 1996.

[15] Y. Iwanaga, *Duality over Auslander-Gorenstein rings*, Math. Scand. 81(2) (1997) 184-190.

[16] O. Iyama, *Symmetry and duality on n-Gorenstein rings*, J. Algebra 269 (2003) 528-535.

[17] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, 217, AMS, 2003.

[18] H. Matsumura, *Commutative Ring Theory*, Cambridge stud. Adv. Math. 8, Cambridge Univ. Press, 1986.

[19] J. Matijevic and P. Roberts, *A conjecture of Nagata on graded Cohen-Macaulay rings*, J. Math. Kyoto Univ., 14 (1974) 125-128.

[20] C. Năstăsescu and F. Van Oystaeyen, *Graded Ring Theory*, Math. Library 28, North Holland, 1982.

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