LINEARITY DEFECTS OF FACE RINGS

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This is a joint work with Kohji Yanagawa, and most results given in this talk are found in [3].

Let $A := \bigoplus_{i \in \mathbb{N}} A_i$ be a polynomial ring $S := K[x_1, \cdots, x_n]$ or an exterior algebra $E := \bigwedge \langle y_1, \cdots, y_n \rangle$ over a field K, $^*\mathrm{mod}\, A$ the category of finitely generated graded (left) A-modules, and $P_\bullet$ a minimal graded free resolution of $M \in {}^*\mathrm{mod}\, A$. The *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$ is the chain complex such that $\mathrm{lin}(P_\bullet)_i = P_i$ and its differential map is given by erasing all the terms of degree ≥ 2 from the matrices of those of $P_\bullet$. According to [2], we call

$$\mathrm{ld}_A(M) := \sup\{\, i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0\}$$

the *linearity defect* of M.

Set $[n] := \{1, 2, \cdots, n\}$ and let Δ be a simplicial complex, i.e., a subset of $2^{[n]}$ with the property that $F \subset G, G \in \Delta$ implies $F \in \Delta$. We set

$$I_\Delta := (\prod_{i \in F} x_i \mid F \subset [n],\, F \notin \Delta\,) \subset S, \quad J_\Delta := (\bigwedge_{i \in F} y_i \mid F \subset [n],\, F \notin \Delta\,) \subset E.$$

The quotient ring $K[\Delta] := S/I_\Delta$ is called the *Stanley-Reisner ring*, and $K\langle \Delta \rangle := E/J_\Delta$ is called the *exterior face ring*. Both are very important in Combinatorial Commutative Algebra.

The following are main results of our talk.

Theorem 1. *For a simplicial complex Δ on $[n]$, we have*

(1) $\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta])$ *(henceforth we set* $\mathrm{ld}(\Delta) := \mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta])$*);*

(2) *if $\Delta \neq 2^T$ for any $T \subset [n]$, $\mathrm{ld}(\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee := \{\, F \subset [n] \mid [n] \setminus F \notin \Delta\,\}$.*

As for $\mathrm{ld}_E(K\langle\Delta\rangle)$, Herzog-Römer and Yanagawa showed the following upper bound: for a simplicial complex Δ on $[n]$, we have

$$\mathrm{ld}_E(K\langle\Delta\rangle) \leq \max\{1,\, n-2\}.$$

In particular, by Theorem 1 we have $\mathrm{ld}(\Delta) \leq n - 2$, if $n \geq 3$. Thus it is natural to ask which simplicial complex attains the equality $\mathrm{ld}(\Delta) = n - 2$. The next is the answer.

Theorem 2. *If $n \geq 4$, we have $\mathrm{ld}(\Delta) = n - 2 \iff \Delta$ is an n-gon (a triangulation of a circle S^1 with n vertices).*

References

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[4] K. Yanagawa, Castelnuovo-Mumford regularity for complexes and weakly Koszul modules, J. Pure and Appl. Algebra 207 (2006), 77–97.