FINITE-TYPE INVARIANTS OF WORDS FOR CURVES

NOBORU ITO

ABSTRACT. This study defines finite-type invariants for curves on surfaces and provides the construction of these finite-type invariants for stable homeomorphism classes of curves on compact oriented surfaces without boundaries. The invariants in this theory are developed using the word theory proposed by Turaev.

1. INTRODUCTION.

V. A. Vassiliev developed a method for knot classification by applying the singularity theory to knots. This method attempts to classify knots by using finite-type invariants (Vassiliev invariants) that are elements of $H^0(\mathcal{K} \setminus \Sigma)$ for the functional space $\mathcal{K}$, which comprises all the smooth mappings of S^1 into $\mathbf{R}^3$, including the set Σ of those mappings that are not embeddings. It remains unknown whether finite-type invariants can be used for the complete classification of knots (the Vassiliev conjecture). V. I. Arnold developed invariants for generic plane curves using a theory similar to that used by Vassiliev. Arnold constructed first-order invariants that are elements of $H^0(\mathcal{F} \setminus \Sigma)$ for the functional space $\mathcal{F}$, which comprises all smooth mappings of S^1 into $\mathbf{R}^2$, including the set Σ of mappings that are not non-generic immersions. M. Polyak and O. Viro presented an explicit formula for second- or third-order Vassiliev invariants by utilizing the Gauss diagram. Polyak also reconstructed Arnold's invariants by using the Gauss diagram in a similar manner and combinatorially defined the finite-type invariants of plane curves. V. Turaev suggested that words and their topology can be considered as generalized objects of curves or knots.

In this study, the author constructs a family of finite-type invariants for stable homeomorphism classes of curves on compact oriented surfaces without boundaries. These invariants are constructed using the word theory proposed by Turaev.

2. CURVES.

A *curve* is a smooth immersion of an oriented circle into a closed oriented surface. The author now defines some types of curves. First, a curve is said to be *generic* if it has only transversal double points of self-intersection. A curve is said to be *singular* if it has only transversal double points, self-tangency points, and triple points of self-intersection. A pointed curve is defined as a curve with a base point marked along the curve except on the self-intersections. Two curves are said to be stably homeomorphic if their regular neighbourhoods are homeomorphic in the

The author is a Research Fellow of the Japan Society for the Promotion of Science. This work was partly supported by KAKENHI. .

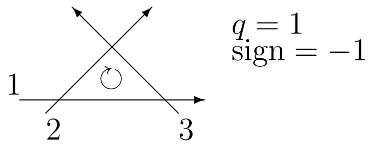


FIGURE 1. Sign of triangle.

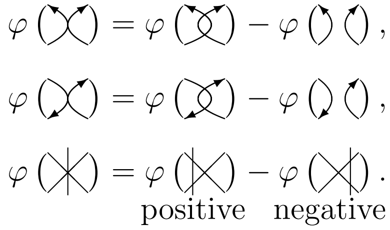
ambient surfaces that map the first curve onto the second one without changing the orientation of the curves and the surfaces. Similarly, two pointed curves are termed as stably homeomorphic if the locations of the marked base points are preserved after the curves undergo stable homeomorphism.

3. Definition of finite-type invariants for immersed curves.

Self-tangency points or triple points can be termed as *singular points.* In particular, a self-tangency point is known as a *direct self-tangency* point if the two tangent branches are obtained using the same tangent vector; otherwise, it is called an inverse self-tangency point. The direction of the resolution of the self-tangency point is said to be *positive* if the resolution generates the part with a larger number of double points. The direction of the resolution of the triple point is said to be *positive* if the resolution generates a part with a *positive triangle*, defined as the number of sides whose orientations coincide with the orientation of the curve by q, where the orientation of the triangle determined by a cyclic order of sides derived from three preimages of the triple point. The sign of the triangle is defined as $(-1)^q$ (Fig. 1, cf. [2]). The direction of the resolution of the singular point is said to be negative if the direction is non-positive. In the case of singular curves, it is possible to resolve singular points away from a marked base point.

Let φ denote every invariant of the surface isotopy classes of generic curves. The value of φ at a singular point is defined as the difference of the values on the curves between the positive and negative resolutions of the singular point whenever three curves are the same, except in the neighbourhood of a point where they are as shown in each of (1), (2) and (3):

(1)
(2)
(3)



Every invariant of the curves is extended inductively to that of singular curves by resolving the singular points using (1), (2), and (3).

Definition 3.1. An invariant of generic curves φ is said to be *a finite-type invariant of order less than or equal to n* if φ vanishes on every singular curve with at least $n+1$ singular points (self-tangency points or triple points), where φ is expanded by (1), (2) and (3).

4. Signed words.

Definition 4.1. Let τ be the involution: $2j \mapsto 2j-1$ and $2j-1 \mapsto 2j$ for every $j \in \mathbf{N}$. Let $\mathcal{A}$ be the set comprising $\{X_j, \overline{X}_j \ (j \in \mathbf{N}) \mid X_j := \tau(2j), \overline{X}_j := 2j, \overline{\overline{X}}_j = X_j\}$. A *signed word of length* $2n$ is a mapping from $\hat{n} := \{1, 2, 3, \ldots, 2n\}$ to $\mathcal{A}$ where each element of $w(\hat{n})$ is the image of precisely two elements of $\hat{n}$ and $w(\hat{n})$ has not both X_j and $\overline{X}_j$ for every $j \in \mathbf{N}$. A signed word of length 0 is denoted by $\emptyset$. For every signed word w of length $2n$, the elements of $w(\hat{n})$ are termed as *letters* of the signed word w. For every letter A of a signed word w, set $sign_w A = -1$ if $A = \overline{X}$ and $sign_w A = 1$ if $A = X$. Two signed words w and w' of length $2n$ are *isomorphic* if there is a bijection $f : \mathcal{A} \to \mathcal{A}$ such that $w' = fw$ and $sign_{w'} w'(i) = sign_{fw} fw(i)$ for every $i \in \hat{n}$. The isomorphism of two signed words w and w' is denoted by $w \simeq w'$. For example, $\overline{X}_1 X_2 X_2 \overline{X}_1 \simeq \overline{X}_5 X_2 X_2 \overline{X}_5$. However, $\overline{X}_1 X_2 X_2 \overline{X}_1$ is not isomorphic to $X_1 \overline{X}_2 \overline{X}_2 X_1$. For two signed words u and w, $u \prec w$ implies that u is a subword of w.

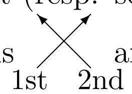
5. Construction of the invariant.

For two arbitrary signed words u and w, $\langle \ , \ \rangle$ can be defined by

$$\langle u, w \rangle = \sum_{v \prec w} (u, v), \tag{4}$$

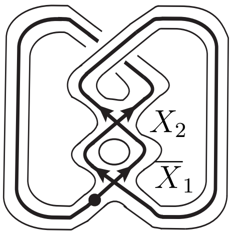
where (u, v) is 1 if $u \simeq v$ and is 0 otherwise. Let k be a field; $\mathbf{W}$, the k-linear space generated by all the isomorphic classes of the signed words; and $\mathbf{W}^*$, the dual space of $\mathbf{W}$. We expand $\langle \ , \ \rangle$ linearly to $\langle \ , \ \rangle : \mathbf{W} \times \mathbf{W} \to k$.

For an arbitrary generic curve Γ, which has m double points, let w_Γ be a signed word: $\hat{m} \to \mathcal{A}$ that is determined by selecting an arbitrary marked base point in the following manner as in [8].

For a given generic curve Γ, which has m double points, used to define w_Γ, first, the double points of Γ are labelled by using distinct letters $A_1, A_2, \ldots, A_{2m}$ such that $A_i = \overline{X}_i$ or X_i. Then, beginning with a marked base point and moving along Γ until returning to the base point, the first instances of all the double points are labelled. Let the i-th double point be labelled A_i. Let t_i^1 (resp. t_i^2) be the tangent vector to Γ at the first (resp. second) passage through this double point. Set $A_i = \overline{X}_i$ if the pair (t_i^1, t_i^2) is 

and $A_i = X_i$ if otherwise. Since every double point is traversed twice, this yields a signed word w_Γ, which is represented by m elements in $\mathcal{A}$.

Definition 5.1. The ν-shift is defined as $\nu(AxAy) = x\overline{A}y\overline{A}$, where $A \in \mathcal{A}$ and x and y are subwords. The *cyclic equivalence* $\sim$ is defined as $w \sim w'$ if and only if there exists $l \in \mathbf{N}$ such that $\nu^l(w) = w'$. Let $\mathbf{W}_n$ be the k-linear space generated by all the isomorphic classes with signed words of length $2n$ and $\mathbf{W}_n^*$ be the dual space of $\mathbf{W}_n$. Let the subspace $\mathbf{W}_n^\nu$ of $\mathbf{W}_n$ be the vector space generated by $\{w \in \mathbf{W}_n \mid \nu(w) = w\}$. Let $(\mathbf{W}_n^\nu)^*$ be the dual space of $\mathbf{W}_n^\nu$. For every cyclic equivalence class containing a signed word v, the sum of all the representative elements of the cyclic equivalence class is denoted by $[v]$. For two arbitrary α and $\beta \in k$ and two arbitrary u and $v \in \mathbf{W}$, let $[\alpha u + \beta v] := \alpha[u] + \beta[v]$.

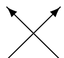
$\overline{X}_1 X_2 \overline{X}_1 X_2 \leftrightarrow$ 

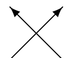
$\cup$ two disks.

FIGURE 2. Signed words and pointed curves.

For example, if the cyclic equivalence of the isomorphic class of the signed words is $\{X_1X_1\overline{X}_2\overline{X}_2,\ \overline{X}_1\overline{X}_2\overline{X}_2\overline{X}_1,\ \overline{X}_2\overline{X}_2X_1X_1,\ X_2X_1X_1X_2\}$ and $v = \overline{X}_2\overline{X}_2X_1X_1$, then $[v] = X_1X_1\overline{X}_2\overline{X}_2 + \overline{X}_1\overline{X}_2\overline{X}_2\overline{X}_1 + \overline{X}_2\overline{X}_2X_1X_1 + X_2X_1X_1X_2$.

We denote the linear space generated by all stable homeomorphism classes of curves on compact oriented surfaces without boundaries by $\mathcal{C}$.

Remark 5.1. By using $\overline{X} \mapsto$ 

and $X \mapsto$ 

every signed word determines a regular neighbourhood of a curve Γ on a surface $\mathcal{S}$, where Γ gives the CW-decomposition of $\mathcal{S}$ (Fig. 2). We denote the linear space generated by all stable homeomorphism classes of curves with n double points on compact oriented surfaces without boundaries by $\mathcal{C}_n$. There exists a bijective mapping from $\mathcal{C}_n$ to $\mathbf{W}_n^\nu$; this has been proved by V. Turaev [8]. In the rest of this paper, we identify $\mathcal{C}_n$ with $\mathbf{W}_n^\nu$ and $\mathcal{C}$ with $\mathbf{W}^\nu$, where $\mathbf{W}^\nu$ is the vector space generated by $\{w \in \mathbf{W} |\ \nu(w) = w\}$.

For a given generic curve Γ, let w_Γ be a signed word that is determined by Γ. For two arbitrary natural numbers m and n, we define a *signed curve invariant of order* n, $SCI_n : \mathbf{W}_m^\nu \to (\mathbf{W}_n^\nu)^*$, as follows:

$$SCI_n(\Gamma) = \langle [\cdot], w_\Gamma \rangle : \mathbf{W}_n \to k. \tag{5}$$

By the definition of SCI_n, we use the following notation:

$$SCI_n(\Gamma)(v) = \langle [v], w_\Gamma \rangle \quad (v \in \mathbf{W}_n). \tag{6}$$

By using the definition of [ ] and $\langle\ ,\rangle$, SCI_n is independent of the choice of the marked base point on the curve Γ.

Theorem 5.1. *SCI_n is a finite-type invariant of order less than or equal to n for an arbitrary generic curve on a surface.*

Let m_w be the number of representative elements of the cyclic equivalence class that contains a signed word w. Let $\overline{w} = \frac{1}{m_w}[w] \in \mathbf{W}^\nu$. Then, $SCI_n(\Gamma) = \langle [\cdot], \overline{w}_\Gamma \rangle$. For two arbitrary signed words v and u, v^* denotes a linear mapping such that $v^*(u) = (v, u)$. SCI_n restricted on $\mathbf{W}_n^\nu$ gives the isomorphism between $\mathbf{W}_n^\nu$ and $(\mathbf{W}_n^\nu)^*$, sending $\overline{w}_\Gamma$ to $[w_\Gamma]^*$. We denote this isomorphism by ι_n.

Theorem 5.2. *For $SCI_k : \mathbf{W}_n^\nu \to (\mathbf{W}_k^\nu)^*$, we establish the following relation:*

$$\frac{(n-k+1)!}{(n-l-1)!(k-l)!} SCI_l = SCI_l \circ \iota_k^{-1} \circ SCI_k \tag{7}$$

$$(1 \leq l \leq k \leq n).$$

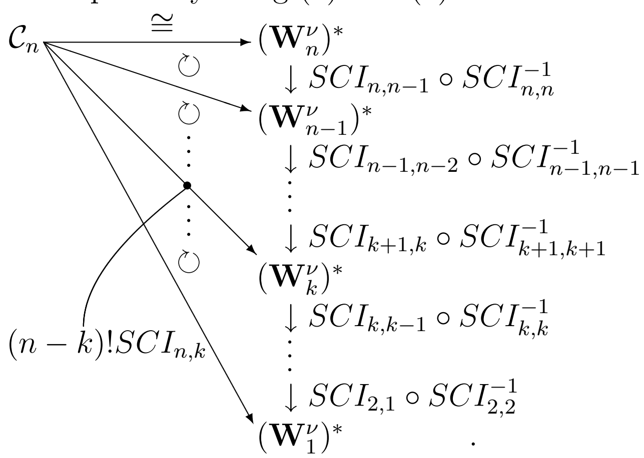
$$(n-k+1)SCI_{k-1} = SCI_{k-1} \circ \iota_k^{-1} \circ SCI_k \ (2 \leq k \leq n) \tag{8}$$

leads to (7).

6. Summary

The author is unable to determine the structure of the linear space $\mathbf{V}_n$ generated by finite-type invariants of order less than or equal to n.

However, for the subspace $(\mathbf{W}_n^\nu)^*$ of $\mathbf{V}_n$, $SCI_{m,l}$ $(1 \leq m, l \leq n)$ have the following relation between the subspaces by using (5) and (8).



(9)

This diagram shows how $SCI_{m,l}$ $(1 \leq m, l \leq n)$ reduce information from $\mathcal{C}_n \cong (\mathbf{W}_n^\nu)^*$ to $(\mathbf{W}_1^\nu)^*$.

References

[1] V. I. Arnold, *Plane curves, their invariants, perestroikas and classifications*, Adv. Sov. Math **21**, Amer. Math. Soc., Providence (1994), 33–91.

[2] V. I. Arnold, *Topological invariants of plane curves and caustics*, University Lecture Series **5**, Providence, RI, 1994.

[3] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[4] J. S. Birman and X. -S. Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), 225–270.

[5] Y. Ohyama, *Vassiliev invariants and local moves of knots*, Rokko Lectures in Mathematics **15**, 2003 (in Japanese) available at http://www.math.kobe-u.ac.jp/publications/rlm15.pdf.

[6] M. Polyak, *Invariants of curves and fronts via Gauss diagrams*, Topology **37** (1998), 989–1009.

[7] V. Turaev, *Curves on surface, charts, and words*, Geom. Dedicata **116** (2005), 203–236.

[8] V. Turaev, *Knots and words*, Int. Math. Res. Not. **2006** (2006), Article ID 84098, 23pp.

[9] V. Turaev, *Lectures on topology of words*, Jpn. J. Math. **2** (2007) 1–39.

[10] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its applications, Adv. Sov. Math. **1**, Amer. Math. Soc., Providence, RI (1990), 23–69.

Department of Pure and Applied Mathematics Waseda University. Tokyo 169-8555, Japan.
E-mail address: noboru@moegi.waseda.jp