The Glauberman-Watanabe corresponding blocks of finite groups with normal defect groups

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Abstract

Harris proved that there is an indecomposable bimodule with a trivial source which induces a Morita equivalence between Glauberman-Watanabe corresponding block algebras of finite groups with normal defect groups and the Glauberman correspondence of characters in corresponding blocks. We note an implication of the Puig correspondence in the context of the Glauberman-Watanabe correspondence, from which Harris’s theorem follows.

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Throughout this article, let $p$ be a prime, and let $(\mathcal{K}, \mathcal{O}, k)$ be a $p$-modular system where $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field $k$ of characteristic $p$ and having a quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article. Let $\mathcal{R} \subseteq \{\mathcal{O}, k\}$.

By a module, we mean an $\mathcal{R}$-free finitely generated left module. For finite groups $G$ and $G'$, an $(\mathcal{R} G, \mathcal{R} G')$-bimodule $M$ can be identified with an $\mathcal{R}[G \times G']$-module by $(g, g') \cdot m = g \cdot m \cdot g'^{-1}$ where $g \in G$, $g' \in G'$ and $m \in M$, and, under this, we can say that $X$ is a trivial source module.

By a character, we always mean an ordinary character over $\mathcal{K}$. Denote by $\text{Irr}(G)$ the set of all irreducible characters of $G$ (or of $\mathcal{K} G$).

By a $(p)$-block (idempotent) of $G$ (over $\mathcal{R}$) or $\mathcal{R} G$, we mean a primitive idempotent of the center $Z(\mathcal{R} G)$ of the block algebra $\mathcal{R} G$ of $G$ over $\mathcal{R}$. The set $\text{Bl}_\mathcal{O}(G)$ of blocks of $\mathcal{O} G$ and the set $\text{Bl}_k(G)$ of blocks of $k G$ corresponds bijectively by the canonical epimorphism $\mathcal{O} G \rightarrow k G$, $x \mapsto x (x \in \mathcal{O} G)$. The set of blocks of $G$ with defect group $D$ is denoted by $\text{Bl}_\mathcal{K}(G|D)$. For $b \in \text{Bl}_\mathcal{O}(G)$, denote by $\text{Irr}(b)$ the subset $\{ \phi | \phi(b) \neq 0 \}$ of $\text{Irr}(G)$, whose element is called a character in a block $b$. Then we have a partition of the characters of $G$ with respect to $p$: $\text{Irr}(G) = \sqcup_{b \in \text{Bl}_\mathcal{O}(G)} \text{Irr}(b)$.

Denote by $\mathcal{Z}\text{Irr}(b)$ the character group of $b$, that is, the free group with the basis set $\text{Irr}(b)$. The map $I : \mathcal{Z}\text{Irr}(b) \rightarrow \mathcal{Z}\text{Irr}(b')$ between character groups where $b' \in \text{Bl}_\mathcal{O}(G')$ is called isometry if it is an isomorphism and preserves the usual inner product, that is, $I(\phi) \in \text{Irr}(b')$ or $-I(\phi) \in \text{Irr}(b')$.

When a finite group $S$ acts on $G$, denote by $G^S$ the centralizer $C_G(S)$ of $S$ in $G$, by $\text{Irr}(G)^S$ the set of all $S$-invariant irreducible characters of $G$, and by $\text{Bl}_\mathcal{K}(G)^S (\text{Bl}_\mathcal{K}(G|D)^S)$ the set of all $S$-invariant blocks of $G$ (with defect group $D$).

Denote $H \trianglelefteq G$ ($H \trianglelefteq G$), when $H$ is a (normal) subgroup of $G$. We use $\Delta$ to indicate diagonal subgroups, that is, $\Delta D = \{(d, d) \in G \times G' | d \in D\}$ for a “common” subgroup $D$ of the two groups $G$ and $G'$. 

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We recall the correspondences of Glauberman (Theorem 2.1) and Watanabe (Theorem 2.3) and results by Koshitani-Michiler and Harris (Theorem 2.4) on the normal defect group case of the Glauberman-Watanabe correspondence. Then we state our main theorem (Theorem 2.5), which is stated and proved in terms of Puig’s theory (see Section 3). For the proof, see [22].

Glauberman gives in [6] some correspondence of characters of finite groups, called the Glauberman correspondence (of characters):

**Theorem 2.1.** (Glauberman [6]) For any pair $(G, S)$ where $G$ is a finite group and $S$ is a finite solvable group acting on $G$ such that $(|G|, |S|) = 1$, there exists a uniquely determined bijective map $\pi(G, S) : \text{Irr}(G)^S \to \text{Irr}(G^S)$ satisfying the following conditions:

(i) For $T \leq S$, $\text{Irr}(G)^S$ is mapped bijectively to $\text{Irr}(G^T)^S$ by $\pi(G, T)$.

(ii) In the situation of (i), $\pi(G, S) = \pi(G^T, S/T) \circ \pi(G, T)$.

(iii) If $S$ is a $q$-group for some prime $q$, then, for $\phi \in \text{Irr}(G)^S$, $\pi(G, S)(\phi)$ is a unique constituent of $\phi|_{G^S}$ with a multiplicity $m_\phi$ not divisible by $q$.

In fact, there exists a sign $\epsilon_\phi \in \{1, -1\}$ such that $m_\phi \equiv \epsilon_\phi \pmod{q}$.

Below, we always assume the following (the action of $S$ on $G$ will be fixed, and the semi-direct product $G \rtimes S$ will be the one defined by this action):

**Condition 2.2.** $G$ is a finite group. $S$ is a finite solvable group acting on $G$ such that $(|G|, |S|) = 1$.

Watanabe began in [24] a block-theoretical study of the Glauberman correspondence and gave a block correspondence induced by the Glauberman correspondence under the condition that the concerned block has a defect group centralized by $S$:

**Theorem 2.3.** (Watanabe [24]) Let $b \in \text{Bl}_G(G|D)^S$ with $D \leq G^S$. Then:

1. All characters in $b$ are $S$-invariant: $\text{Irr}(b) = \text{Irr}(b)^S$.
2. There exists $w(b) \in \text{Bl}_G(G|D)$ such that $\text{Irr}(w(b)) = \{\pi(G, S)(\phi) \mid \phi \in \text{Irr}(b)\}$.
3. Brauer categories (see [23, Section 47]) of $b$ and $w(b)$ are equivalent.
4. There is a perfect isometry between $\text{ZIrr}(b)$ and $\text{ZIrr}(w(b))$ induced by the Glauberman correspondence, which is a composition of the following perfect isometries: Let $1 = S_0 \leq S_1 \leq S_2 \leq \cdots \leq S_n = S$ be a composition series of $S$ such that $|S_i/S_{i-1}|$ is a prime for $1 \leq i \leq n$ and $w_i(b)$ a unique block of $G^{S_i}$ determined by (2) for $S_i$. Then there is a perfect isometry $\text{ZIrr}(w_{i-1}(b)) \simeq \text{ZIrr}(w_i(b))$ mapping $\phi_{i-1} \in \text{Irr}(w_{i-1}(b))$ to $\epsilon_{\phi_{i-1}} \pi(G^{S_{i-1}}, S_i/S_{i-1})(\phi_{i-1})$, where $\epsilon_{\phi_{i-1}}$ is the sign described in Proposition 2.1(iii) (under suitable choices when $|S_i/S_{i-1}| = 2$).
The perfect isometry in (4) gives an isotypy between $b$ and $w(b)$.

The correspondence of blocks over $\mathcal{R}$ determined by Theorem 2.3(2) is called the Glauberman-Watanabe correspondence (of blocks).

In particular, if $S$ centralizes a Sylow $p$-subgroup of $G$, then the Glauberman-Watanabe correspondence induces a bijective correspondence between $\text{Bl}_{\mathcal{R}}(G)^S$ and $\text{Bl}_{\mathcal{R}}(G^S)$ such that corresponding blocks have a common defect group.

For the notions perfect isometry and isotypy, see [3]. We only say that perfect isometry (isotypy) is a phenomenon in the character level which is often said to be a “shadow” of a (splendid) derived equivalence. In fact, it is known that if there is a two-sided tilting complex (such that every term consists of a direct sum of trivial source modules with “diagonal” vertices) inducing a derived equivalence between block algebras over $\mathcal{O}$, then the isometry between character groups of concerned blocks induced by that equivalence gives a perfect isometry (isotypy) ([3], [20]).

Generally, in $p$-block theory of finite groups, “some good relation” between blocks having equivalent Brauer categories, is expected.

Although blocks having equivalent Brauer categories is not necessarily (splendid) derived equivalent, we may expect, due to Theorem 2.3(4) (Theorem 2.3(5)), a (splendid) derived equivalence between Glauberman-Watanabe corresponding blocks such that the Glauberman correspondence is induced.

When defect groups are normal, it is shown that the isotypy in Theorem 2.3(5) is in fact a “shadow” of a splendid derived equivalence. Koshitani and Michler proved in [10] that Glauberman-Watanabe corresponding block algebras over $k$ with normal defect groups are Morita equivalent. In fact, Koshitani noted in [9] that they are Puig equivalent, which in particular implies that Glauberman-Watanabe corresponding block algebras over $\mathcal{R}$ with normal defect groups are Morita equivalent and splendid derived equivalent. Then, Harris showed in [7] the following, which implies Theorem 2.3(5) in the normal defect group case with the consideration of the signs $\epsilon_\phi$ for $\phi \in \text{Irr}(b)$:

**Theorem 2.4.** (Harris [7]) Let $b \in \text{Bl}_{\mathcal{O}}(G|D)^S$ with $D \leq G^S$ and $D \leq G$. Then:

1. (Dade, Koshitani, Michler [10] [9]) The Glauberman-Watanabe corresponding blocks are Puig equivalent.

2. There is an indecomposable trivial source $(\mathcal{O}G, \mathcal{O}G^S)$-bimodule inducing a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^Sw(b)$ and inducing the Glauberman correspondence between $\text{Irr}(b)$ and $\text{Irr}(w(b))$.

The following Theorem 2.5, which in particular says that the Glauberman correspondence is induced by the “multiplication by the idempotent $l_i$”, is our main result and gives an alternative proof of Theorem 2.4:

**Theorem 2.5.** ([22]) Let $b \in \text{Bl}_{\mathcal{O}}(G|D)^S$ with $D \leq G^S$ and $D \leq G$. Then, with the notations in Theorem 2.3(4) (in particular, $w_0(b) = b$, $w_n(b) = w(b)$),
there exist pointed groups $G_{(b)}=G^S_{β_0}, \ G^{S_1}_{β_1}, \ G^{S_2}_{β_2}, \ \ldots, \ G^{S_n}_{β_n}=G^S_β$ of $OGb$ satisfying the following, for any $i$:

(i) $G_{(b)}=G^{S_0}_{β_0} ≥ G^{S_1}_{β_1} ≥ G^{S_2}_{β_2} ≥ \ldots ≥ G^{S_n}_{β_n}=G^S_β$

(ii) $β_i$ is the unique point of $(OG)^{S_i}$ such that the multiplicity $m(β_i, β_{i-1})$ of $β_i$ in $β_{i-1}$ is not divisible by $|S_i/S_{i-1}|$.

(iii) $G^{S_i}_{β_i}$ has a defect pointed group $D_δ$ where $δ$ is any point of $(OG)^D_i$.

In particular, $OGb$ and $(OGb)_{β_i}$ have common source algebras.

(iv) $(OGb)_{β_i} \simeq OG^{S_i}w_i(b)$ as interior $G^{S_i}$-algebras. In fact, for $l_i∈β_i$, the map, $y_i \mapsto y_i- l_i$ where $y_i ∈ OG^{S_i}w_i(b)$, is an isomorphism of interior $G^{S_i}$-algebra from $OG^{S_i}w_i(b)$ to $l_iOG l_i$, and so $l_iOG l_i = OG^{S_i}w_i(b)l_i$.

In particular, $OGb$ and $OG^{S_i}w_i(b)$ are Puig equivalent.

(v) For $l_i∈β_i$, the $(OG^{S_i}, OG)$-bimodule $l_iOG$ (which is a trivial source $O[G^{S_i}×G]$-module) induces a Morita equivalence between $OGb$ and $OG^{S_i}w_i(b)$ and the Glauberman correspondence between $Irr(b)$ and $Irr(w_i(b))$.

Remark 2.6. A block algebra with a normal defect group is Morita equivalent to some twisted group algebra (Külshammer[11], Puig[16]), and hence Morita equivalence classes are determined by the groups and the 2-cocycles which describe the twisted group algebras. In [10] a Morita equivalence between $kGb$ and $kG^S w(b)$ is shown by reducing the problem to the problem comparing the 2-cocycles corresponding to $kGb$ and $kG^S w(b)$. (Then Dade’s theorem guarantees the equivalence of the 2-cocycles, see [10, Lemma 3.2].)

On the other hand, the Morita equivalence between $OGb$ and $OG^S w(b)$ in Theorem 2.5 is given by firstly constructing some primitive interior $G^S$-algebra (determined by the combination of the Glauberman correspondence and the Puig correspondence (see (3A) below), see Proposition 4.1) which is Morita equivalent to $OGb$ by the construction, and secondly showing that it is isomorphic to $OG^S w(b)$ as a primitive interior $G^S$-algebra. (For our proof of the isomorphism, Puig’s result (3B) and (3C) in Section 3 are crucial.)

As the usual study of the Glauberman correspondence, we use induction for the proof of Theorem 2.5. That is, by (i) and (ii) in Theorem 2.1, it suffices to consider the following case:

Condition 2.7. $S$ is a cyclic group of prime order $q$ (such that $q \nmid |G|$).

Below, (2.7) will be always assumed. Then the results in Section 5 can be applied for $G^{S_{i-1}}, S_i/S_{i-1}$ and $w_{i-1}(b)$ in place of $G, S$ and $b$ there, and Theorem 2.5 follows from the induction on $i$. Note that $lOG$ in Section 5 corresponds to the $(OG^{S_i}, OG^{S_{i-1}})$-bimodule $l_iOG l_{i-1}$ where $l_i∈β_i$ and $l_{i-1}∈β_{i-1}$. 
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In this section, we collect terminologies and facts in G-algebra theory used in this article. We mainly cite from [23] rather than original articles [15] and [16].

For an $\mathcal{R}$-algebra $A$, a conjugacy class of primitive idempotents of $A$ are called a point of $A$.

An $\mathcal{R}$-algebra $A$ is called a G-algebra if $G$ acts on $A$ via $\mathcal{R}$-algebra automorphism of $A$. A homomorphism of $G$-algebra is an $\mathcal{R}$-algebra homomorphism which commutes with the actions of $G$. Denote by $A^H$ the subalgebra of $A$ consisting of all the fixed elements of $A$ by the action of $H \leq G$. If $1_A$ is the unique idempotent of $A^G$, $A$ is called a primitive G-algebra. Below, let $A$ be a G-algebra.

A pair $(H, \beta)$ of a subgroup $H$ of $G$ and a point $\beta$ of $A^H$, usually denoted by $H_\beta$, is called a pointed group of $A$. $G$ acts on pointed groups of $A$ by $(H_\beta)^g := H^{g}\beta$ for $g \in G$ where $\beta^g$ is a point of $A^{H^g}$ containing $\beta^g$ for $l \in \beta$. For $l \in \beta$, $lA_l$ has the obvious $H$-algebra structure. $A_{\beta}$ is the primitive $H$-algebra isomorphic to $lA_l$, called the localization of $A$ with respect to $H_\beta$.

Let $K_\sigma$ and $H_\beta$ be pointed groups of $A$ such that $K \succeq H$. The multiplicity of $P$ in $\sigma$, denoted by $m(\beta, \sigma)$, is the number of the occurrence of the idempotents belonging to $\beta$ in a primitive idempotent decomposition of $j \in \sigma$ in $A^H$. Denote $K_\sigma \geq H_\beta$, if $m(\beta, \sigma) \neq 0$. Denote $K_\sigma pr H_\beta$ if $\sigma \subseteq \text{Tr}_{H_\beta}^K(A^H, \beta A^H)$ where \text{Tr}_{H_\beta}^K(a) = \sum_{g\in[H:K]} a^g$ for $a \in A^H$. The relations $\succeq$ and $pr$ make the set of pointed groups of $A$ a partially ordered set, respectively.

A pointed group $P_\gamma$ of $A$ is called a local pointed group if $P_\gamma$ is “minimal” with respect to the relation $pr$, that is, there is no pointed group $Q_\delta$ such that $P_\gamma \supset P_\delta$, in which case $P$ is necessarily a p-group, see [23, Lemma 14.4].

When $P_\gamma$ is a “maximal” (with respect to the relation $\succeq$) local pointed group such that $H_\beta \succeq P_\gamma$, then $P_\gamma, P, i(\in \gamma)$ and $A_\gamma$ are called a defect pointed group, a defect group, a source idempotent and a source algebra, of $H_\beta$ (or $\beta$), respectively. Source algebras of $H_\beta$ are Morita equivalent to $A_{\beta}$, see [23, Proposition 18.10]. Defect pointed groups (and hence defect groups) of $H_\beta$ are determined up to $H$-conjugation, see [23, Corollary 18.6]. When $A$ is primitive, a defect (pointed) group and a source algebra of the pointed group $G_{\{1, A\}}$ is called a defect (pointed) group and a source algebra of $A$, respectively.

The simple k-algebra $A(P_\gamma) = A^P/M_\beta$, where $M_\beta$ is the unique maximal ideal of $A^P$ not containing $\gamma$, has an $N_G(P_\gamma)/P$-algebra structure, where $N_G(P_\gamma) = \{g \in G | (P_\gamma)^g = P_\gamma\}$. The unique simple $A(P_\gamma)$-module $V(P_\gamma)$ can be endowed with a $k_2N_G(P_\gamma)/P$-module structure, where $k_2N_G(P_\gamma)/P$ is some twisted group algebra of $N_G(P_\gamma)/P$ over $k$ determined by the Skolem-Noether theorem, see [23, Example 10.4]. When $A$ is primitive with a defect pointed group $P_\gamma$, the $k_2N_G(P_\gamma)/P$-module $V(P_\gamma)$ is called a defect multiplicity module of $A$, which is always indecomposable projective module, see [23, Theorems 19.2]. The following correspondence is called the Puig correspondence (see [23, Theorems 19.1]):
(3A) There is a one-to-one correspondence between the set of points of $A^H$ having a defect pointed group $P_\gamma$ and the set of isomorphism classes of indecomposable direct summands of $V(P_\gamma) \big|_{k\text{Aut}_H(P_\gamma)/P}$.

An $\mathcal{R}$-algebra $B$ is called an interior $G$-algebra if there is a group homomorphism $\varphi$ from $G$ to the unit group $B^\times$ of $B$. The structural homomorphism $\varphi$ can be extended to $\varphi : \mathcal{R}G \to B$. A homomorphism of interior $G$-algebras is an $\mathcal{R}$-algebra homomorphism which commute with the structural homomorphisms. $B$ is also considered as a $G$-algebra whose structural action of $G$ on $B$ is given by the conjugation of $\varphi(g)$ ($g \in G$). The localization $B_\beta$ of $B$ with respect to a pointed group $H_\beta$ of $B$ is considered with the obvious interior $H$-algebra structure. If $B$ is a primitive interior $G$-algebra (that is, primitive as a $G$-algebra), then there is a unique block $b$ of $G$ such that $\varphi(b)1_B = 1_B$, and in this case, we say that $B$ is in $b$, see [18, 4.1].

Example 3.1. For an $\mathcal{R}G$-module $L$, $\text{End}_\mathcal{R}(L)$ is an interior $G$-algebra with the structural map $\varphi : G \to \text{Aut}_\mathcal{R}(L)$, $g \mapsto m_g$, where $m_g$ is the automorphism induced by the left action of $g \in G$ on $L$.

The set of points of $\text{End}_\mathcal{R}(L)^H = \text{End}_{\mathcal{R}H}(L)$ corresponds bijectively to the the set of the isomorphism classes of the indecomposable direct summands of $L|_H^G$ by $\beta \mapsto l(L)$ where $l \in \beta$. For pointed groups $K_\sigma$ and $H_\beta$ of $\text{End}_\mathcal{R}(L)$, $K_\sigma \geq H_\beta$ if and only if $l(L)$ is isomorphic to a direct summand of $j(L)|_H^K$, see [23, Example 13.4], and $K_\sigma \triangleright H_\beta$ if and only if $j(L)$ is isomorphic to a direct summand of $l(L)|_H^K$, see [23, Proposition 17.11], where $j \in \sigma$ and $i \in \beta$.

Below, let $L$ be an indecomposable $\mathcal{R}G$-module, that is, $\text{End}_\mathcal{R}(L)$ be a primitive interior $G$-algebra.

$P$ is a defect group of $\text{End}_\mathcal{R}(L)$ if and only if $P$ is a minimal subgroup of $G$ such that $L|_P|_P^G$, that is, a vertex of $L$, see [23, Theorem 17.2]. For a pointed group $P_\gamma$ of $\text{End}_\mathcal{R}(L)$, $P_\gamma$ is a defect pointed group of $\text{End}_\mathcal{R}(L)$ if and only if $P$ is a vertex of $L$ and $L$ is isomorphic to a direct summand of $i(L)|_P^G$, where $i \in \gamma$, that is, $i(L)$ is a source $\mathcal{R}P$-module of $L$, see [23, Proposition 18.11]. (A module whose source is a trivial module $\mathcal{R}$ is called a trivial source module.)

$L$ is called simply defective if $\text{End}_\mathcal{R}(L)$ has a simple defect multiplicity module. Simple $kG$-modules, full $OG$-lattices of irreducible characters of $G$, see [15, Proposition 1.6] or [23, p.213], and $\mathcal{R}G$-modules with “full” vertex, see [1, Proposition 1.2], are simply defective.

Example 3.2. Group algebra $\mathcal{R}G$ is an interior $G$-algebra with the structural map $G \to (\mathcal{R}G)^\times$, $g \mapsto g$. The set of points of $(\mathcal{R}G)^G = Z(\mathcal{R}G)$ corresponds to the set of blocks of $G$.

For $b \in \text{Bl}_\mathcal{R}(G)$, the block algebra $\mathcal{R}Gb$ is a primitive interior $G$-algebra with the structural map $g \mapsto gb$. A defect group, a source idempotent, of the block $b$ are those of $\mathcal{R}Gb$. Defect multiplicity modules of $\mathcal{R}Gb$ are simple modules, see [23, Corollary 37.6].

When $b$ has a normal defect group $D$, the structure of the source algebra is known ([16]). In particular, the following holds: Let $D_b$ be a defect
pointed group and \(i \in \delta\), and for \(g \in N_G(D_\delta)\) let \(a_g \in (i(RGb)i)^{\times}\) be such that \(a_g^{-1}(ai)a_g = u^g i\) for any \(u \in D\) (for the existence of \(a_g\), see [23, Proposition 44.2]). Then (see [23, Theorem 44.3]):

\[(3B)\] As an \((RD,RD)\)-bimodule, \(i(RGb)i = iRGi\) decomposes as \(\oplus_g RD a_g\) where \(g\) runs over a coset representaives of \(DC_G(D)\) in \(N_G(D_\delta)\).

In fact, an \(R\)-algebra \(iRGi\) is isomorphic to some twisted group algebra over \(D \rtimes (N_G(D_\delta)/DC_G(D))\), see [23, Theorem 45.12].

For another block \(b'\) of a finite group \(G'\), it is called that \(RGb\) and \(RG'b'\) (or \(b\) and \(b'\)) are Puig equivalent, if \(b\) and \(b'\) have an isomorphic defect group \(D\) and the source algebras of \(RGb\) and \(RG'b'\) are isomorphic as interior \(D\)-algebras. It is known, see [17, Lemma 7.8] and [19], that block algebras over \(k\) are Puig equivalent if and only if block algebras over \(O\) are Puig equivalent if and only if there is a trivial source module inducing a Morita equivalence between concerned block algebras.

An \(R\)-algebra \(C\) is said to be \(iRGbi\)-algebra if \(C\) is endowed with a unitary \(R\)-algebra homomorphism \(\rho\) from \(iRGbi\) to \(C\) where \(i\) is a source idempotent of \(b \in \text{Bl}_R(G)\), see [18, 4.2]. The homorphism of \(iRGbi\)-algebras is an \(R\)-algebra homomorphism commuting with the structural homomorphisms. \(C\) is primitive if \(1_C\) is primitive in the centralizer of the image of \(iRGbi\) in \(C\). Then (see [18, Proposition 4.3]):

\[(3C)\] There is an equivalence between the category of the primitive interior \(G\)-algebras in \(b\) and the category of the primitive \(iRGbi\)-algebras such that an object \(B\) of the former corresponds to an object \(\rho(i)B\rho(i)\) of the latter.

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In this section, we note an implication of the Puig correspondence in the context of the Glauberman-Watanabe correspondence. Recall our assumptions (2.2) and (2.7).

Since, as is well-known, the modules over twisted group algebra over a group \(N\) can be viewed as modules over the ordinary group algebra of the group \(\tilde{N}\) which is obtained by some central extension of \(N\), and simple projective \(k\tilde{N}\)-modules can be uniquely lifted to the projective \(O\tilde{N}\)-modules which is a full \(O\tilde{N}\)-lattice of some irreducible character of \(\tilde{N}\), we can “apply the Glauberman correspondence to the defect multiplicity module” of the primitive interior \(G\)-algebras as in Proposition 4.1 (see also [8] and [12] for the adaptation of the Glauberman correspondence for the modules over twisted group algebra). Combining this with the Puig correspondence (3A), we see:

**Proposition 4.1.** Let \(A\) be a primitive interior \(G\)-algebra over \(R\) which can be extended to an interior \(G \times S\)-algebra \(\tilde{A}\) and which has an \(S\)-centralized defect group \(P\) and a simple defect multiplicity module. Let \(\alpha = \{1_A\}\) be a unique point of \(A^G\). Then:
There is a unique point $\beta$ of $A^G$ satisfying the following:

(i) The pointed group $G^S_\beta$ of $A$ has $P$ as a defect group.

(ii) $q \nmid m(\beta, \alpha)$.

In fact, $m(\beta, \alpha) \equiv \pm 1 \pmod q$ holds.

(2) ([2, Proposition 4.8]) $G^G_\alpha \pr G^G_\beta$.

(3) If the block belonging to $A$ has an $S$-centralized defect group, then $A_\beta$ belongs to the Glauberman-Watanabe corresponding block.

For example, applying Proposition 4.1 to the situation of Example 3.1, we can see the following:

**Corollary 4.2.** Assume that $S$ centralizes a Sylow $p$-subgroup of $G$ and let $P$ be an $S$-centralized $p$-subgroup of $G$. Then:

(1) If $X$ is an $S$-invariant simply defective indecomposable $RG$-module with vertex $P$, then there exists, unique up to isomorphism, an indecomposable direct summand $X'$ of $X|_{G^S}$ satisfying the following:

(i) $X'$ has $P$ as a vertex.

(ii) The multiplicity $m(X', X)$ of $X'$ in $X|_{G^S}$ is not divisible by $q$.

In fact, $X'$ is simply defective and $m(X', X) \equiv \pm 1 \pmod q$.

(2) If $Y$ is a simply defective indecomposable $RG^S$-module with vertex $P$, then there exists, unique up to isomorphism, an $S$-invariant indecomposable direct summand $Y''$ of $Y|_{G^S}$ satisfying the following:

(i) $Y''$ has $P$ as a vertex.

(ii) The multiplicity $n(Y'', Y)$ of $Y''$ in $Y|_{G^S}$ is not divisible by $q$.

In fact, $Y''$ is simply defective and $n(Y'', Y) \equiv \pm 1 \pmod q$.

(3) The correspondences given in (1) and (2) are mutually inverse, that is, $(X')'' \simeq X$ and $(Y'')' \simeq Y$. Moreover, $m(X, X') = n(X, X')$.

(4) There is a bijective correspondence between the set of the isomorphism classes of $S$-invariant simply defective indecomposable $RG$-modules with vertex $P$ and the set of the isomorphism classes of simply defective indecomposable $RG^S$-modules with vertex $P$. The set of isomorphism classes of source $RP$-modules of the corresponding modules is same. The corresponding modules belong to the Glauberman-Watanabe corresponding blocks.
In this section, we apply Proposition 4.1 to the situation of Example 3.2, and state that the condition that an $S$-invariant block algebra $A = \mathcal{O}Gb$ has an $S$-centralized normal defect group is a sufficient condition for a primitive interior $G^S$-algebra $A_\beta$ as in Proposition 4.1 being a Glauberman-Watanebe corresponding block algebra (in this situation, the simple modules in $kG^b$ and $kG^Sw(b)$ correspond bijectively by the correspondence as in Corollary 4.2).

We collect the conditions and notations always assumed below (the groups $G$ and $S$ are described in (2.2) and (2.7)):

**Condition 5.1.** $b$ is an $S$-invariant block of $G$ having a defect group $D$ centralized by $S$. $\beta$ is a unique point of $(\mathcal{O}Gb)^G$ (determined by Proposition 4.1 applied to $A = \mathcal{O}Gb$) such that $G^S_\beta$ has a defect group $D$ and $q \not| m(\beta, \alpha)$ where $\alpha = \{b\}$ is the unique point of $(\mathcal{O}Gb)^G$. Let $l \in \beta$ and $m = m(\beta, \alpha)$.

Considering $\mathcal{O}Gb$ as an $(\mathcal{O}G, \mathcal{O}G)$-bimodule, we can view $\operatorname{End}_{\mathcal{O}[1 \times G]}(\mathcal{O}Gb)$ as an interior $G$-algebra using the left $\mathcal{O}G$-module structure of $\mathcal{O}Gb$. Then we have a well-known isomorphism of interior $G$-algebras

$$\mathcal{O}Gb \rightarrow \operatorname{End}_{\mathcal{O}[1 \times G]}(\mathcal{O}Gb), \quad x \mapsto \varphi_x \quad (5A)$$

where $\varphi_x$ is the $\mathcal{O}$-endomorphism of $\mathcal{O}Gb$ given by the left action of $x \in \mathcal{O}Gb$. By the identification (5A), for $H \leq G$, the set of points of $(\mathcal{O}Gb)^H$ with defect group $D$ corresponds to the isomorphism classes of indecomposable direct summands of $\mathcal{O}Gb|_{H \times G}$ with a vertex $\Delta D$.

**Proposition 5.2.** $\mathcal{O}Gb|_{G \times G}^{G \times G}$ has, unique up to isomorphism, an indecomposable direct summand $M$ satisfying the following:

(i) $M$ has $\Delta D$ as a vertex.

(ii) The multiplicity $m_M$ of $M$ in $\mathcal{O}Gb|_{G \times G}^{G \times G}$ is not divisible by $q$.

In fact, $m_M = m$ (hence $m_M \equiv \pm 1 \pmod{q}$), and $M$ is isomorphic to $l(\mathcal{O}Gb) = l\mathcal{O}G$ (hence is an $(\mathcal{O}G^S \mathcal{O}w(b), \mathcal{O}Gb)$-bimodule).

In the normal defect case of our setting, since the Puig correspondence does not lose the information ([5, Proposition 2.12]), we see the following:

**Corollary 5.3.** Assume $D \leq G$. Then any indecomposable direct summand of $\mathcal{O}Gb|_{G \times G}^{G \times G}$ has $\Delta D$ as a vertex. Hence, we have the following indecomposable direct sum decomposition of $\mathcal{O}Gb|_{G \times G}^{G \times G}$:

$$\mathcal{O}Gb|_{G \times G}^{G \times G} \simeq m(l\mathcal{O}G) \oplus (\oplus_i m_i M_i),$$

where $m_i \equiv 0 \pmod{q}$ for all $i$ (and note that $m \equiv \pm 1 \pmod{q}$).
We can show that there is some defect pointed groups $D_δ, D_{δ'}$ of $OGb$ and $OGS\psi(b)$ respectively such that $i'\equiv li'\equiv i$ for some $i, i' \in δ, δ'$. Using (3B), we see that $i'\psi(OGS\psi(b))i'\rightarrow i'(lOGb)i'\equiv iOGb i$ ($x\rightarrow lx$) gives an isomorphism of the primitive $i'(OGS\psi(b))i'$-algebras. Then from (3C) we have:

**Theorem 5.4.** Assume $D_δ G$. Then $(OGb)_δ$ and $OGS\psi(b)$ are isomorphic as interior $G_S$-algebras. In particular, $OGb$ and $OGS\psi(b)$ are Puig equivalent ([9],[7]).

Since $lOG$ is a direct summand of $OGb_{\psi}\otimes_{G_S} lOG$, $lOG$ has a trivial source. By Theorem 5.4, tensoring $lOG$ over $OG$, that is, multiplication by the idempotent $l$, induces a Morita equivalence between $OGb$ and $OGS\psi(b)$. Moreover, $lOG$ induces the Glauberman correspondence (also “for Brauer characters”), since the restriction of $OGb$-modules to $OGS$-modules is induced by $OGb_{\psi}\otimes_{G_S} lOG$, $OGb_{\psi}\otimes_{G_S} lOG$ has a description as in Corollary 5.3, and the Glauberman correspondence has a property as described in Theorem 2.1(iii).

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We assume (2.2), (2.7) and (5.1) in this section too.

Firstly, we set the notations. Let $\lambda$ be a generator of the dual group $S=\text{Hom}(S, K)$ of $S$. $S$ acts on $\text{Irr}(G \times S)$, that is, for $\psi \in \text{Irr}(G \times S)$, the function $\lambda \psi$ defined by $\lambda \psi(gs)=\lambda(s)\psi(gs)$ is a character of $G \times S$ where $\lambda \in S, g \in G$ and $s \in S$. In our situation, $b$ is covered by $q$ distinct blocks of $G \times S$, that is, $b=\sum_{i=0}^{q-1} b_i$ for blocks $b_i$ of $G \times S$. For $\phi \in \text{Irr}(b)$, there is an extension of $\phi$ to the character $\phi_0$ of $G \times S$ satisying $\phi_0(\psi GS) \epsilon_\phi = \epsilon_\phi \pi(G, S)(\phi)$ where $c \in G$, $s(\neq 1) \in S$ and $\epsilon_\phi$ is the sign described in Theorem 2.1(iii). Then $\{\phi_0 \mid \phi \in \text{Irr}(b)\}$ forms the set of the characters in some block of $G \times S$ covering $b$ (under appropriate choices of the signs $\epsilon_\phi$ when $q=2$), which is denoted by $b_0$ and called in [24, p.555] a canonical extension of $b$. (It is unique when $q$ is odd. On the other hand, there are two canonical extensions when $q=2$.) Then, for $1' \in S$ and $t \in \mathbb{Z}$, $\{\phi_0 \mid \phi \in \text{Irr}(b_0)\}$ forms the set of characters in some block of $G \times S$ covering $b$, denoted by $b_t$. Elements of $S$ can be viewed as elements of $\text{Irr}(S)$, and let $c_t$ be the block of $GS$ corresponding to $1' \in \text{Irr}(S)$ for $t \in \mathbb{Z}$. Note that $b_t=b_{t'} (c_t=c_{t'})$ if and only if $t\equiv t'$ mod $q$.

Since the Glauberman correspondence induces an isotypy between Glauberman-Watanabe corresponding blocks, it is desirable to exist a two-sided complex inducing a splendid derived equivalence and the Glauberman correspondence. In fact, in the normal defect group case, this is proved, see Theorem 2.6.

On the other hand, Okuyama pointed out in [14] that there is some pairwise orthogonal (possibly zero) idempotents $b_i, 0 \leq i \leq q-1$, of $(OGb)_{\psi}$ such that $b=\sum_{i=0}^{q-1} b_i$ and, as generalized characters of $K[G^S \times G]$, 

$$\Phi_0 - \Phi_r = \sum_{\phi \in \text{Irr}(b)} \epsilon_\phi (\pi(G, S)(\phi) \times \phi) \quad \text{for} \quad 1 \leq r \leq q-1, \quad (6A)$$

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where \( \Phi_i \) is the character corresponding to \( b_i \mathcal{K}G \), \( \hat{\phi} \) is the dual of \( \phi \) and \( \epsilon_\phi \) is the sign described in Theorem 2.1(iii). In fact, we can take as

\[
b_i = \sum_{j=0}^{q-1} e_j \hat{b}_{j+i} \quad \text{for} \quad 0 \leq i \leq q - 1,
\]

see [21, Section 4]. (When \( q = 2 \), \( b_0 \) depends on the choice of \( \hat{b}_0 \).) From this discription of \( b_i \), we see

\[
b_i \mathcal{O}G \cong \epsilon_{t-i} \mathcal{O}(G \times S) \hat{b}_i \mathcal{G}^{S \times G}.
\]

Using the \((\mathcal{O}G^S, \mathcal{O}G)\)-bimodule isomorphism (6C), we can show a more precise statement of Proposition 5.2 (where for a group \( H \) and an \( \mathcal{O}H \)-module \( \mathcal{M} \), we denote by \( \mathcal{M}_P \) a maximal direct summand of \( \mathcal{M} \) any of whose indecomposable direct summand has \( P \) as a vertex):

**Proposition 6.1.** With the above notations, we have the following isomorphisms of \((\mathcal{O}G^S, \mathcal{O}G)\)-bimodules: when \( q \) is odd, if \( m \equiv 1 \) (mod \( q \)) then

\[
(b_0 \mathcal{O}G)_{\Delta D} \cong \mathcal{L} \mathcal{O}G \oplus (b_r \mathcal{O}G)_{\Delta D}
\]

and if \( m \equiv -1 \) (mod \( q \)) then

\[
(b_r \mathcal{O}G)_{\Delta D} \cong \mathcal{L} \mathcal{O}G \oplus (b_0 \mathcal{O}G)_{\Delta D},
\]

and when \( q = 2 \), depending on the choice of \( b_0 \), we have (6.1.1) or (6.1.2). Hence, if moreover we assume \( D \leq G \), then we have

\[
b_0 \mathcal{O}G \cong \mathcal{L} \mathcal{O}G \oplus b_r \mathcal{O}G \quad \text{or} \quad b_r \mathcal{O}G \cong \mathcal{L} \mathcal{O}G \oplus b_0 \mathcal{O}G.
\]

Therefore, the normal defect case fits into the above observation. That is, there is a complex \( C^* \) of \((\mathcal{O}G^S, \mathcal{O}G)\)-bimodules which induces a splendid derived equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^S w(b) \) and whose character is “canonically” reduced to the left hand side of \((*)\) (hence which induces the Glauberman correspondence). In fact, by Proposition 6.1, for example, we can take

\[
C^* : \cdots \rightarrow 0 \rightarrow b_r \mathcal{O}G \rightarrow b_0 \mathcal{O}G \rightarrow 0 \rightarrow \cdots
\]

where the degree of \( b_0 \mathcal{O}G \) is 0 and the non-trivial differential is induced by inclusion in the case \( m \equiv 1 \) (mod \( q \)) and projection in the case \( m \equiv -1 \) (mod \( q \)). Note that \( C^* \cong \mathcal{L} \mathcal{O}G \) or \( C^* \cong \mathcal{L} \mathcal{O}G[1] \) in the appropriate derived (or homotopy) category where we view \( \mathcal{L} \mathcal{O}G \) as a complex concentrated in degree 0.

**Remark 6.2.** From (6A) and Proposition 6.1, we see that, when \( D \leq G \), the character of \( \mathcal{K} \otimes \mathcal{O} \mathcal{L} \mathcal{O}G \) is \( \sum_{\phi \in \text{Irr}(b)} \pi(G, S)(\phi) \times \hat{\phi} \). Hence, we have another proof of the fact that \( \mathcal{L} \mathcal{O}G \) induces a Morita equivalence between \( \mathcal{O}Gb \) and \( \mathcal{O}G^S w(b) \) by utilizing (6A) and (6B), since the bimodule which induces a one-to-one correspondence between characters belonging to two blocks and which is projective as left and right modules induces a Morita equivalence ([4, Théorème 0.2]).
References