有限グラフに付随するエッジ環および
そのトーリックイデアルのイニシャルイデアルの深度

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Introduction

This report is a summary of the papers [3] and [4].

Let G be a finite simple graph, i.e., a finite graph without loops and multiple edges, on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_r\}$ its edge set. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ be the polynomial ring in d variables over a field K and $K[G]$ the subring of $K[\mathbf{t}]$ generated by $\mathbf{t}^e = t_i t_j$ with $e = \{i, j\} \in E(G)$, that is, $K[G] = K[\mathbf{t}^e : e \in E(G)] \subset K[\mathbf{t}]$. The semigroup ring $K[G]$ is called the *edge ring* of G. Let $K[\mathbf{x}] = K[x_1, \ldots, x_r]$ be the polynomial ring in r variables over a field K. We define the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, r$. We call the kernel I_G of π the *toric ideal* of G. Then one has $K[G] \cong K[\mathbf{x}]/I_G$.

Now, it is well known that if G is connected and nonbipartite (resp. bipartite), then $\text{Krull-dim}\, K[G] = d$ (resp. $\text{Krull-dim}\, K[G] = d - 1$), where $\text{Krull-dim}\, K[G]$ denotes the Krull dimension of $K[G]$. On the other hand, $\operatorname{depth} K[G]$ is presumably unknown, where $\operatorname{depth} K[G]$ denotes the depth of $K[G]$. Hence, it is reasonable to investigate the depths of edge rings.

We consider the normality of edge rings. The edge ring $K[G]$ is called *Cohen–Macaulay* if its depth coincides with its Krull dimension. In general, if a semigroup ring is normal, then it is also Cohen–Macaulay. The criterion of normality of edge rings is known by [6, Corollary 2.3] due to Ohsugi and Hibi. It follows from the criterion that $K[G]$ is normal if either G is bipartite or $d \leq 6$. In addition, if $d = 7$, then there are six finite connected graphs whose edge rings are nonnormal, while all of them are Cohen–Macaulay, i.e., their depths are 7. From these facts together with our computational experiments for $d \geq 8$, we propose the following

Conjecture 0.1. Let G be a finite connected nonbipartite graph on $[d]$ with $d \geq 7$. Then $\operatorname{depth} K[G] \geq 7$.

We refer the reader to [2, Chapter 2] for fundamental materials on Gröbner bases. Let $<$ be a monomial order on $K[\mathbf{x}]$ and $\operatorname{in}_<(I_G)$ the initial ideal of I_G with respect to $<$. In general, the inequality $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G) \leq \operatorname{depth} K[\mathbf{x}]/I_G (= \operatorname{depth} K[G])$ holds. Thus, in the case where $K[G]$ is normal, we consider $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G)$, in other words, how $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G)$ decreases from $\operatorname{depth} K[G]$. Similarly to Conjecture 0.1, our computational experiences yield the following

Conjecture 0.2. Let G be a finite connected nonbipartite graph on $[d]$ with $d \geq 6$ and suppose that its edge ring $K[G]$ is normal. Then $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G) \geq 6$ for any monomial order $<$ on $K[\mathbf{x}]$.

Even though Conjectures 0.1 and 0.2 are completely open, by taking these into consideration, we prove the following theorems:

Theorem 0.3. *Given integers f and d with $7 \leq f \leq d$, there exists a finite connected graph G on $[d]$ with* $\operatorname{depth} K[G] = f$ *and* $\text{Krull-dim}\, K[G] = d$.

Theorem 0.4. *Given integers f and d with $6 \leq f \leq d$, there exists a finite connected nonbipartite graph G on $[d]$ together with a reverse lexicographic order $<_{\text{rev}}$ on $K[\mathbf{x}]$ and a lexicographic order $<_{\text{lex}}$ on $K[\mathbf{x}]$ such that*

(i) *$K[G]$ is normal;*
(ii) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G) = f$;
(iii) *$K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$ is Cohen–Macaulay.*

Let $k \geq 1$ be an arbitrary integer and G_{k+6} the finite graph on $[k+6]$ of Figure 0.1. The essential part of a proof of Theorem 0.3 is to show that

$$\operatorname{depth} K[G_{k+6}] = \operatorname{depth} K[\mathbf{x}]/I_{G_{k+6}} = 7.$$

We will prove this equality in Section 1.

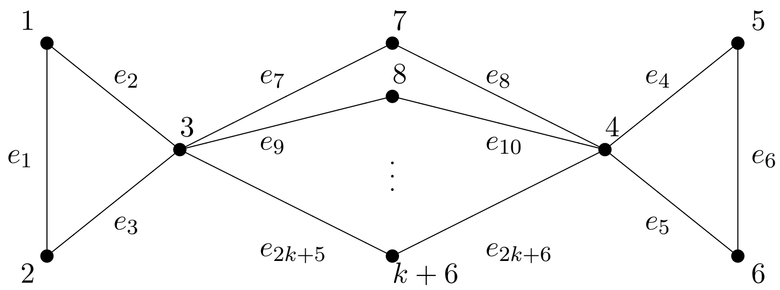

Figure 0.1. (finite graph G_{k+6})

Once we know that $\operatorname{depth} K[G_{k+6}] = 7$, to prove Theorem 0.3 is straightforward. In fact, given integers f and d with $7 \leq f \leq d$, let Γ denote the finite graph G_{d-f+7} on $[d-f+7]$ and write G for the finite graph on $[d]$ obtained from Γ by adding $f-7$ edges $\{1, d-f+8\}, \{1, d-f+9\}, \ldots, \{1, d\}$ to Γ. It then follows that $\operatorname{depth} K[G] = \operatorname{depth} K[\Gamma] + f - 7$. Since $\operatorname{depth} K[\Gamma] = 7$, one has $\operatorname{depth} K[G] = f$, as required.

On the other hand, for an arbitrary integer $k \geq 1$, we introduce the graph H_{k+5} on $[k+5]$ drawn in Figure 0.2. Clearly, the edge ring $K[H_{k+5}]$ is normal from the criterion of the normality. (Moreover, even if we add some new edges to H_{k+5}, the normality is invariable.) The graph H_{k+5} plays an important role in our proof of Theorem 0.4. The essential step in the proof of Theorem 0.4 is to show that

$$\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_{H_{k+5}}) = 6$$

and $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{lex}}}(I_{H_{k+5}})$ is Cohen–Macaulay, where $<_{\mathrm{rev}}$ (resp. $<_{\mathrm{lex}}$) denotes the reverse lexicographic order (resp. the lexicographic order) on $K[\mathbf{x}] = K[x_1, \dots, x_{2k+5}]$ induced by the ordering $x_1 > \cdots > x_{2k+5}$ of the variables. We will prove these in Section 2.

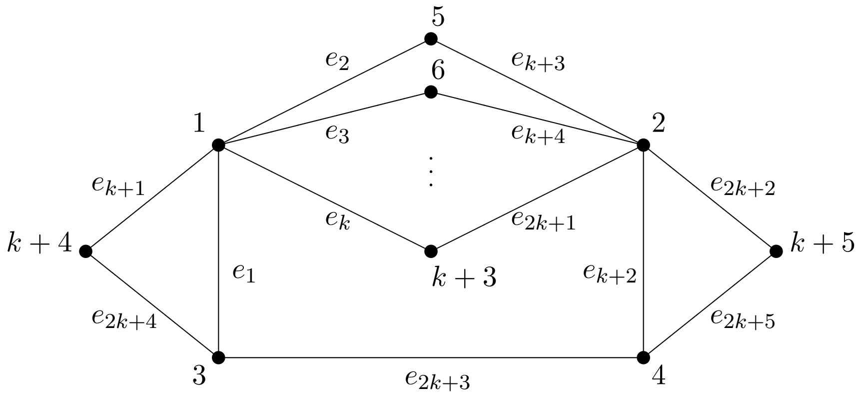

Figure 0.2. (finite graph H_{k+5})

Once we establish the above statements, to prove Theorem 0.4 is straightforward by the similar way to Theorem 0.3.

1. The depth of the edge ring of G_{k+6}

Let $G = G_{k+6}$ of Figure 0.1. In this section, we prove that $\operatorname{depth} K[G] = 7$. The first subsection is devoted to showing that $\operatorname{depth} K[G] \leq 7$ and the second one is devoted to showing that $\operatorname{depth} K[G] \geq 7$.

1.1. **Proof of** $\operatorname{depth} K[G] \leq 7$**.** The Auslander–Buchsbaum formula says that

$$\operatorname{depth} K[G] + \operatorname{pd} K[G] = \operatorname{depth} K[\mathbf{x}],$$

where $\operatorname{pd} K[G]$ denotes the projective dimension of $K[G]$. Since the number of edges of G is $r = 2k+6$, one has $\operatorname{depth} K[\mathbf{x}] = 2k+6$. Thus, in order to prove that $\operatorname{depth} K[G] \leq 7$, we may show that $\operatorname{pd} K[G] \geq 2k-1$.

For an edge $e_i = \{i_1, i_2\} \in E(G)$, we set $\mathbf{a}_i = \mathbf{e}_{i_1} + \mathbf{e}_{i_2} \in \mathbb{Z}^d$, where $\mathbf{e}_j$ is the jth unit vector of $\mathbb{R}^d$. Let S_G be the semigroup arising from G, i.e., $S_G = \mathbb{N}\mathbf{a}_1 + \cdots + \mathbb{N}\mathbf{a}_r$. Then we regard $K[G]$ as an S_G-graded semigroup ring. Let $\operatorname{pd} K[G] = \max\{i : \beta_{i,\mathbf{a}}(K[G]) \neq 0\}$, where $\beta_{i,\mathbf{a}}(K[G]) = \dim_K \operatorname{Tor}_i(K[G], K)_{\mathbf{a}}$ is the ith Betti number of $K[G]$ in degree $\mathbf{a}$. To show $\operatorname{pd} K[G] \geq 2k-1$, it is sufficient to prove that $\beta_{2k-1,\mathbf{a}}(K[G]) \neq 0$ for some $\mathbf{a} \in S_G$.

For $\mathbf{a} \in S_G$, we define the simplicial complex

$$\Delta_{\mathbf{a}} = \{F \subset [r] \ : \ \mathbf{a} - \mathbf{n}_F \in S_G\},$$

where $\mathbf{n}_F = \sum_{l \in F} \mathbf{a}_l$. We use the following theorem due to Briales, Campillo, Marijuán, and Pisón [1].

Lemma 1.1 ([1, Theorem 2.1])**.** *Let G be a finite simple graph. Then*

$$\beta_{j+1,\mathbf{a}}(K[G]) = \dim_K \tilde{H}_j(\Delta_{\mathbf{a}}; K).$$

Let us consider the simplicial complex $\Delta_{\mathbf{a}}$ with

$$\mathbf{a} = (1, 1, k+1, k+1, 1, 1, 2, 2, \dots, 2) \in S_G.$$

Then we can show that

$$\dim_K \tilde{H}_{2k-2}(\Delta_{\mathbf{a}}; K) \neq 0.$$

Therefore, it follows from Lemma 1.1 that $\operatorname{pd} K[G] \geq 2k-1$, as desired.

1.2. **Proof of** $\operatorname{depth} K[G] \geq 7$**.** Since

$$\operatorname{depth} K[G] = \operatorname{depth} K[\mathbf{x}]/I_G \geq \operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G),$$

(see, e.g., [2, Theorem 3.3.4 (d)],) it is sufficient to prove that $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G) \geq 7$. Thus, we find a Gröbner basis of I_G to obtain a system of generators of an initial ideal of I_G with respect to $<_{\text{lex}}$.

Set $C_1 = (e_2, e_1, e_3)$ and $C_2 = (e_4, e_6, e_5)$, both of which are 3-cycles of G. Then there are 2 kinds of primitive even closed walks of G:

(1) a 4-cycle: $(e_{2i+5}, e_{2i+6}, e_{2j+6}, e_{2j+5})$, where $1 \leq i < j \leq k$;
(2) a walk on two 3-cycles C_1, C_2 and the length 2 paths combining C_1 and C_2: $(C_1, e_{2p+5}, e_{2p+6}, C_2, e_{2q+6}, e_{2q+5})$, where $1 \leq p \leq q \leq k$.

It was proved in [7, Lemma 3.1] that binomials corresponding to these primitive even closed walks generate the toric ideal I_G.

Lemma 1.2. *The toric ideal I_G is generated by the following binomials:*

$$\begin{array}{ll} x_{2i+5}x_{2j+6} - x_{2i+6}x_{2j+5}, & 1 \leq i < j \leq k, \\ x_1x_4x_5x_{2p+5}x_{2q+5} - x_2x_3x_6x_{2p+6}x_{2q+6}, & 1 \leq p \leq q \leq k. \end{array}$$

Moreover they form a Gröbner basis of I_G with respect to $<_{\text{lex}}$.

From Lemma 1.2, we obtain a system of generators of $\operatorname{in}_{<_{\text{lex}}}(I_G)$, which is the set of the following monomials:

$$\begin{array}{ll} x_{2i+5}x_{2j+6}, & 1 \leq i < j \leq k, \\ x_1x_4x_5x_{2p+5}x_{2q+5}, & 1 \leq p \leq q \leq k. \end{array}$$

Let I' denote the ideal generated by the monomials $x_{2i+5}x_{2j+6}, 1 \leq i < j \leq k$. Then one has

$$\begin{aligned} \operatorname{in}_{<_{\text{lex}}}(I_G) &= x_1x_4x_5(x_7, x_9, \dots, x_{2k+5})^2 + I' \\ &= ((x_7, x_9, \dots, x_{2k+5})^2 + I') \cap ((x_1x_4x_5) + I'). \end{aligned}$$

We set $I_1 = (x_7, x_9, \dots, x_{2k+5})^2 + I'$ and $I_2 = (x_1x_4x_5) + I'$. Then we have $\operatorname{depth} K[\mathbf{x}]/I_1 \geq 7$, $\operatorname{depth} K[\mathbf{x}]/I_2 \geq 7$ and $\operatorname{depth} K[\mathbf{x}]/(I_1 + I_2) \geq 6$. By applying the short exact sequence

$$0 \to K[\mathbf{x}]/I_1 \cap I_2 \to K[\mathbf{x}]/I_1 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(I_1 + I_2) \to 0,$$

we can show that

$$(\operatorname{depth} K[G] \geq) \operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G) = \operatorname{depth} K[\mathbf{x}]/I_1 \cap I_2 \geq 7.$$

2. The depth of the initial ideal of the normal edge ring of H_{k+5}

Let $H = H_{k+5}$. In this section, we prove that $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{rev}}}(I_H) = 6$ and $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{lex}}}(I_H)$ is Cohen–Macaulay. First, we find systems of generators of $\operatorname{in}_{<_{\mathrm{rev}}}(I_H)$ and $\operatorname{in}_{<_{\mathrm{lex}}}(I_H)$ in Subsection 2.1. Next, we show that $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{rev}}}(I_H) \leq 6$ in Subsection 2.2 and $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{rev}}}(I_H) \geq 6$ in Subsection 2.3. Finally, in Subsection 2.4, we prove that $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{lex}}}(I_H)$ is Cohen–Macaulay.

2.1. **Generators of initial ideals.** There are 4 kinds of primitive even closed walks of H:

(1) a 4-cycle: $(e_i, e_{k+1+i}, e_{k+1+j}, e_j)$, where $2 \leq i < j \leq k$;
(2) a walk on two 3-cycles and the same edge e_{2k+3} combining two cycles: $(e_1, e_{k+1}, e_{2k+4}, e_{2k+3}, e_{k+2}, e_{2k+2}, e_{2k+5}, e_{2k+3})$;
(3) a 6-cycle: $(e_r, e_{k+1+r}, e_{k+2}, e_{2k+3}, e_{2k+4}, e_{k+1})$ or $(e_r, e_{k+1+r}, e_{2k+2}, e_{2k+5}, e_{2k+3}, e_1)$, where $2 \leq r \leq k$;
(4) a walk on two 3-cycles and the length 2 paths combining two cycles: $(e_{k+2}, e_{2k+5}, e_{2k+2}, e_{k+1+p}, e_p, e_1, e_{2k+4}, e_{k+1}, e_q, e_{k+1+q})$, where $2 \leq p \leq q \leq k$.

Similarly to the case of $I_{G_{k+6}}$, we obtain a Gröbner basis of I_H with respect to both $<_{\mathrm{rev}}$ and $<_{\mathrm{lex}}$.

Lemma 2.1. *The toric ideal I_H is generated by the following binomials:*

$$\begin{aligned}
&x_i x_{k+j+1} - x_j x_{k+i+1}, && 2 \leq i < j \leq k,\\
&x_1 x_{k+2} x_{2k+4} x_{2k+5} - x_{k+1} x_{2k+2} x_{2k+3}^2, &&\\
&x_r x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+r} x_{2k+3}, && 2 \leq r \leq k,\\
&x_1 x_{k+1+r} x_{2k+5} - x_r x_{2k+2} x_{2k+3}, && 2 \leq r \leq k,\\
&x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5} - x_p x_q x_{k+2} x_{2k+2} x_{2k+4}, && 2 \leq p \leq q \leq k.
\end{aligned}$$

Moreover they form a Gröbner basis of I_H with respect to both $<_{\mathrm{rev}}$ and $<_{\mathrm{lex}}$.

By virtue of Lemma 2.1, we obtain systems of generators of $\operatorname{in}_{<_{\mathrm{rev}}}(I_H)$ and $\operatorname{in}_{<_{\mathrm{lex}}}(I_H)$.

Corollary 2.2. *The initial ideal of I_H with respect to $<_{\mathrm{rev}}$ is generated by the following monomials:*

$$\begin{aligned}
&x_j x_{k+1+i}, && 2 \leq i < j \leq k,\\
&x_{k+1} x_{2k+2} x_{2k+3}^2, &&\\
&x_{k+1} x_{k+1+r} x_{2k+3},\ x_r x_{2k+2} x_{2k+3}, && 2 \leq r \leq k,\\
&x_p x_q x_{k+2} x_{2k+2} x_{2k+4}, && 2 \leq p \leq q \leq k.
\end{aligned}$$

Corollary 2.3. *The initial ideal of I_G with respect to $<_{\mathrm{lex}}$ is generated by the following monomials:*

$$\begin{aligned}
&x_i x_{k+1+j}, && 2 \leq i < j \leq k,\\
&x_1 x_{k+2} x_{2k+4} x_{2k+5}, &&\\
&x_r x_{k+2} x_{2k+4},\ x_1 x_{k+1+r} x_{2k+5}, && 2 \leq r \leq k.
\end{aligned}$$

In particular, $\operatorname{in}_{<_{\mathrm{lex}}}(I_H)$ is a squarefree monomial ideal.

Let I denote the initial ideal of I_H with respect to $<_{\mathrm{rev}}$ and J that with respect to $<_{\mathrm{lex}}$.

2.2. **Proof of** $\operatorname{depth} K[\mathbf{x}]/I \le 6$**.** From the Auslander–Buchsbaum formula, we may prove that $\operatorname{pd} K[\mathbf{x}]/I \ge r-6 = 2k-1$, where $r = 2k+5$ is the number of edges of G. We recall from [5] the fundamental technique to compute the Betti numbers of (non-squarefree) monomial ideals. For a multidegree $\mathbf{a} \in \mathbb{Z}^r_{\ge 0}$, we define

$$\mathbf{K}^{\mathbf{a}}(I) = \{\alpha \in \{0,1\}^r : \mathbf{x}^{\mathbf{a}-\alpha} \in I\}$$

to be the *Koszul simplicial complex* of I in degree $\mathbf{a}$.

Lemma 2.4. ([5, Theorem 1.34]) *Let $S = K[x_1, \ldots, x_r]$ be a polynomial ring over a field K, I a monomial ideal of S and $\mathbf{a} \in \mathbb{Z}^r_{\ge 0}$ a vector. Then*

$$\beta_{i+1,\mathbf{a}}(S/I) = \dim_K \tilde{H}_{i-1}(\mathbf{K}^{\mathbf{a}}(I); K).$$

Set $\mathbf{a} = \sum_{j=2}^{k}(\mathbf{e}_j + \mathbf{e}_{k+1+j}) + \mathbf{e}_{k+1} + \mathbf{e}_{2k+2} + 2\mathbf{e}_{2k+3} \in \mathbb{Z}^{2k+5}_{\ge 0}$, where $\mathbf{e}_i$ is the ith unit vector of $\mathbb{R}^{2k+5}$. Then we can show that

$$\dim_K \tilde{H}_{2k-3}(\mathbf{K}^{\mathbf{a}}(I); K) \ne 0,$$

which means that $\operatorname{pd} K[\mathbf{x}]/I \ge 2k-1$ by virtue of Lemma 2.4.

2.3. **Proof of** $\operatorname{depth} K[\mathbf{x}]/I \ge 6$**.** A proof of $\operatorname{depth} K[\mathbf{x}]/I \ge 6$ is similar to that of $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_{G_{k+6}}) \ge 7$ in Subsection 1.2. We write the ideal I as the intersection of ideals for each of which it is easy to estimate the depth, though the method of division is technical.

2.4. **Cohen–Macaulayness of** $K[\mathbf{x}]/J$**.** Finally, we prove that $K[\mathbf{x}]/J$ is Cohen–Macaulay. Since J is a squarefree monomial ideal, we can regard J as a Stanley–Reisner ideal I_Δ of some simplicial complex Δ. It is known that the Stanley–Reisner ring $K[\Delta] = K[\mathbf{x}]/I_\Delta$ is Cohen–Macaulay if Δ is shellable, which is a special property on simplicial complexes. Our proof is given by showing that Δ is shellable.

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