CLUSTER TILTING MODULES AND QUADRATIC FORMS

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1. Introduction

One of the basic homological invariants of algebras is the global dimension and it can be used to express the complexity of algebras. The algebras with global dimension 0 are semisimple, and their theories were established at the beginning of the 20th century.

The next fundamental classes are algebras with global dimension at most 1, and these classes are characterized as path algebras of quivers. In 1972, Gabriel gave the following famous theorem [G].

Theorem 1.1 (Gabriel)**.** *Let Q be a quiver and q_Q be the quadratic form of Q.*

(1) *The algebra kQ is representation-finite if and only if Q is one of the Dynkin quivers.*
(2) *If Q is a Dynkin quiver, the map $\mathbf{dim} : X \to \mathbf{dim}X$ gives a bijection between the isomorphism classes of indecomposable kQ-modules and the positive roots of q_Q.*

Our aim is to give a generalization of Gabriel's theorem. First let us consider how to generalize the theorem. In Gabriel's theorem we only consider the algebras with global dimension at most 1, so that it is natural to deal with algebras with global dimension at most 2. One of such generalizations was already given by Bongartz [B].

Theorem 1.2 (Bongartz)**.** *Let Λ be a representation-directed algebra with* $\operatorname{gl.dim}\Lambda \leq 2$ *and q_Λ be the quadratic form of Λ. The map $\mathbf{dim} : X \to \mathbf{dim}X$ gives a bijection between the isomorphism classes of indecomposable Λ-modules and the positive roots of q_Λ.*

It is known that any representation-directed algebra is representation-finite. Therefore, if the algebra has global dimension at most 1, the theorem is nothing but Gabriel's one. Bongartz' theorem is far from being true if an algebra is not representation-directed. In this paper we study a different class of algebras from the viewpoint of cluster tilting theory (refer to nice survey papers [A],[BM] and [K]), and instead of considering all indecomposable modules we focus on some important modules.

Conventions and notations. Throughout this paper, we always suppose that k is an algebraically closed field, and that an algebra means a basic indecomposable (connected) finite dimensional k-algebra. We denote by Λ a finite dimensional k-algebra. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules and by $\operatorname{add}M$ the subcategory of $\operatorname{mod}\Lambda$ consisting of direct summands of finite direct sums of M.

2. n-CLUSTER TILTING MODULES AND n-REPRESENTATION-FINITE ALGEBRAS

Definition 2.1. We call a finite dimensional algebra Λ is *n-representation-finite* (for a positive integer n) if $\operatorname{gl.dim}\Lambda \leq n$ and there exists an *n-cluster tilting Λ-module* M, i.e.

$$\begin{aligned} \operatorname{add} M &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M,X) = 0 \text{ for any } 0 < i < n\}, \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X,M) = 0 \text{ for any } 0 < i < n\}. \end{aligned}$$

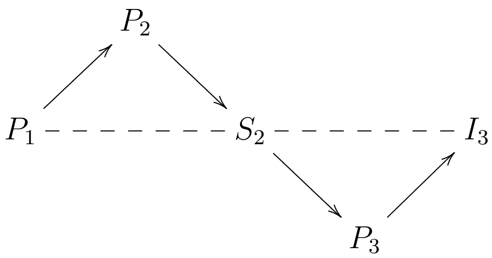
We call a Λ-module X *cluster-indecomposable* if X is indecomposable and it is isomorphic to a direct summand of M.

Remark 2.2. 1-representation-finite algebras are path algebras of a Dynkin quiver. Indeed Λ is 1-representation-finite if and only if there exists a module $M \in \operatorname{mod}\Lambda$ such that $\operatorname{add} M = \operatorname{mod}\Lambda$, so that Λ is 1-representation-finite if and only if it is representation-finite with $\operatorname{gl.dim}\Lambda \leq 1$ if and only if $\Lambda = kQ$ for a Dynkin quiver Q.

Example 2.3. Let Λ be the algebra given by the following quiver with the relation

$$1 \longleftarrow 2 \longleftarrow 3$$

Then there exists a 2-cluster tilting module. Indeed, the AR quiver of $\operatorname{mod}\Lambda$ is the following quiver.



Then it is easy to check that $P_1 \oplus P_2 \oplus P_3 \oplus I_3$ is a 2-cluster tilting module. On the other hand, one can cheak $\operatorname{gl.dim}\Lambda \leq 2$. So the given algebra is 2-representation-finite.

Proposition 2.4. *Let Λ be an n-representation-finite algebra and*

$$\tau_n := D\operatorname{Ext}^n_\Lambda(-,\Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$$

be the functor. Let $P_1, \cdots, P_N$ be the isomorphism classes of indecomposable projective Λ-modules, and let $I_i := \nu P_i$ the corresponding indecomposable injective Λ-modules.

(a) *There exists $\sigma \in \mathfrak{S}_N$ and positive integers $\ell_1, \cdots \ell_N$ such that $\tau_n^{\ell_i - 1} I_i \simeq P_{\sigma(i)}$ for any i.*

(b) *There exists a unique basic n-cluster tilting Λ-module, which is given as a direct sum of the following mutually non-isomorphic indecomposable Λ-modules.*

$$\begin{array}{llllll} I_1, & \tau_n I_1, & \tau_n^2 I_1, & \cdots & \tau_n^{\ell_1-2} I_1, & \tau_n^{\ell_1-1} I_1 \simeq P_{\sigma(1)} \\ I_2, & \tau_n I_2, & \tau_n^2 I_2, & \cdots & \tau_n^{\ell_2-2} I_2, & \tau_n^{\ell_2-1} I_2 \simeq P_{\sigma(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_N, & \tau_n I_N, & \tau_n^2 I_N, & \cdots & \tau_n^{\ell_N-2} I_N, & \tau_n^{\ell_N-1} I_N \simeq P_{\sigma(N)} \end{array}$$

Example 2.5. Let Λ be the algebra given by the following quiver with relations

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & \nearrow & & \searrow & & & \\ & & 5 & - & - & - & 2 & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ 6 & - & - & - & 4 & - & - & - & 1 \end{array}.$$

Then this algebra is 2-representation-finite and we have

$$\begin{gathered} I_1 \cong P_3 \\ I_2 \cong P_5 \\ I_3 \cong P_6 \\ I_4, \tau_2 I_4 \cong P_2 \\ I_5, \tau_2 I_5 \cong P_4 \\ I_6, \tau_2 I_6 \cong S_4, \tau_2^2 I_6 \cong P_1 \end{gathered}$$

Then the cluster tilting module is given as a direct sum of following modules.

$$I_1, I_2, I_3, I_4, I_5, I_6, P_2, P_4, S_4, P_1.$$

Definition 2.6. Let Λ be a finite dimensional algebra of finite global dimension. The quadratic form of Λ is defined as a form

$$q_\Lambda(\mathbf{dim}X) := \sum_{l \geq 0} (-1)^l \dim_k \operatorname{Ext}^l_\Lambda(X, X)$$

for $X \in \operatorname{mod} \Lambda$.

Then using the notion of n-representation-finite, we generalize Theorem 1.1 as follows.

Theorem 2.7. *Let Λ be an n-representation-finite algebra and q_Λ be the quadratic form of Λ. Then for any cluster-indecomposable Λ-module X, the dimension vector* $\mathbf{dim}X$ *gives a positive root of q_Λ uniquely.*

Then it is natural to ask the converse, that is, which roots correspond to cluster-indecomposable modules? Next we consider the characterization of these roots.

Definition 2.8. Let Λ be a finite dimensional algebra with a complete set $\{e_1, ..., e_N\}$ of primitive orthogonal idempotents. The *Cartan matrix* of Λ is the (N, N)-matrix

$$M_\Lambda = \begin{bmatrix} z_{11} & \cdots & z_{1N} \\ \vdots & \cdots & \vdots \\ z_{N1} & \cdots & z_{NN} \end{bmatrix} \in \mathbb{M}_N(\mathbb{Z})$$

where $z_{ji} = \dim_K e_i \Lambda e_j$, for $i, j \in \{1, ..., N\}$.

Moreover, if Λ has finite global dimension n, the *Coxeter matrix* of Λ is defined by the matrix

$$C_\Lambda := (-1)^n M_\Lambda^t M_\Lambda^{-1}.$$

Definition 2.9. Let Λ be a 2-representation-finite algebra. We call a root $\mathbf{x} \in \mathbb{Z}^N$ of q_Λ *cluster-root* if there is a cluster-indecomposable Λ-module X such that $\mathbf{dim}X = \mathbf{x}$. We call $\mathbf{x} \in \mathbb{Z}^N$ *C-positive* if $C_\Lambda^m \mathbf{x} \geq 0$ for any $m \in \mathbb{Z}$.

Then we obtain the following result.

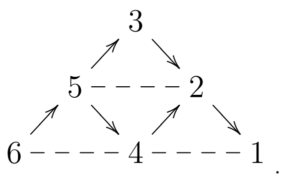
Theorem 2.10. *Cluster-roots are C-positive.*

We expect the converse holds. Namely

Conjecture 2.11. *C-positive roots are cluster-roots.*

We illustrate the considerations by using the following example.

Example 2.12. Let Λ be the algebra given by the following quiver with the relations



Then the quadratic form of Λ is

$$q_\Lambda = \sum_{i \in Q_0} x_i^2 - x_1x_2 - x_2x_3 - x_2x_4 - x_3x_5 - x_4x_5 - x_5x_6 + x_1x_4 + x_2x_5 + x_4x_6.$$

Then the all positive roots of q_Λ are given as follows.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Form these roots, C-positive roots are given as follows

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

and one can check that they are cluster-roots by Example 2.5.

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